Commutative Algebra II Syllabus Extended

Kyle Thompson

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1 First Half

1.1 Existence of primes using Zorns Lemma

We take the set of ideals $\{I \subsetneq R\}$ then for some chain $\{I_i\}$ we have the upper bound $\bigcup I_i$ this is an ideal since each I_i is. So by Zorn there is a maximal ideal. Maximal implies prime is trivial since for $ab \in \mathfrak{m},\ a, b \notin \mathfrak{m}$ then $\mathfrak{m} + (a) = \mathfrak{m} + (b) = R$ so $R = (\mathfrak{m} + (a))(\mathfrak{m} + (b)) = \mathfrak{m}^2 + a\mathfrak{m} + b\mathfrak{m} + (ab) \subset \mathfrak{m}$ so $\mathfrak{m} = R$

1.2 Characterisation $\sqrt{I} = \bigcap_{I \subset \mathfrak{p}} \mathfrak{p}$

For simplicity we look at the case of the nilradical since $x^n \in I \iff \bar{x}^n = 0 \mod I$. So by the correspondance theorem since $\sqrt{I}, \bigcap_{I \subset \mathfrak{p}} \mathfrak{p}$ are both ideals over I they are equal if and only if they are equal under the quotient. So this falls to showing

$$\sqrt{0} = \bigcap_{\mathfrak{p} \in \operatorname{spec} R/I} \mathfrak{p}$$

So for any nilpotent f. We have that $f \in \mathfrak{p}$ since $f^n = 0 \mod \mathfrak{p}$. So f is in the intersection. Then for f non nilpotent we take the set of ideals

$$\Sigma = \{J \subset R/I | f \notin \sqrt{J}\}$$

Clearly this is nonempty since $f \notin \sqrt{0}$. So we have a maximal element say P. We then show that P is prime so f is'nt in the intersection. This argument generalises to finding primes disjoint from any multiplicitively closed set
We do the standard argument. Take $ab \in P, a, b \notin P$ then $f^n \in P + (a), f^m \in P + b$ so $f^{m+n} \in (P + (a))(P + (b)) = P^2 + aP + bP + (ab) \subset P$ so $P \notin \Sigma$ which is a contradiction so P must've been prime

1.3 Krull Dimension of a ring and height of a prime ideal

For a ring the Krull Dimension $\dim(R)$ is defined as the the largest n so that there is a strict chain of prime ideals

$$\mathfrak{p}_0 \subsetneq ... \subsetneq \mathfrak{p}_n$$

We then define the height of a prime $\mathfrak{q}$ as the largest length of chain

$$\mathfrak{p}_0 \subsetneq ... \subsetneq \mathfrak{p}_n = \mathfrak{q}$$

Which by the standard correspondance is just $\dim(R_\mathfrak{p})$

1.4 Localisation $S^{-1}R$ and $S^{-1}M$, its kernel and exactness properties

For a multiplicatively closed set $S \subset R$ we define the localisation of M at S as

$$S^{-1}M = \left\{ \frac{a}{b} | a \in M, b \in S \right\} / \sim$$

Where $x/y \sim a/b \iff \exists s \in S,\ s(bx - ay) = 0$. We define $M_\mathfrak{p}$ for a prime ideal as $(R \setminus \mathfrak{p})^{-1}M$. This is clearly a multiplicitive set since $ab \in \mathfrak{p} \iff a, b \in \mathfrak{p}$ and we take the negation of this. Clearly for M finitely generated. $S^{-1}M = 0 \iff \exists s \in S$ where $s \in \text{ann}(M = (m_1...m_k))$ since $m_i/y \sim 0/1$ so $s_i m_i = 0$ then $\prod s_i \in \text{ann}(M)$. The other direction is trivial. As $sx = 0$ so $x/y \sim 0/1$

Clearly we can extend this construction to a functor, $R - mod \to S^{-1}R - mod$ by sending a map φ to the map $S^{-1}\varphi : m/s \mapsto \varphi(m)/s$. This functor is exact since if we take the exact sequence

$$N \xrightarrow{i} M \xrightarrow{p} Q$$

Then under $S^{-1}-$ we have that the sequence

$$S^{-1}N \xrightarrow{S^{-1}i} S^{-1}M \xrightarrow{S^{-1}p} S^{-1}Q$$

Clearly this satisfies $\operatorname{im} S^{-1}i \subset \ker S^{-1}p$ by functoriality (this is equivalent to saying that the composition factors through the zero map which it does since it did in $R - Mod$ and $S^{-1}0 = 0$). Then for $m/s \in \ker S^{-1}p$ we have that $p(m)/s = 0/1$ so $\tilde{s}p(m) = 0$ but in $S^{-1}R$ $\tilde{s}$ is a unit so $p(m) = 0$ so $\operatorname{im} S^{-1}i = \ker S^{-1}p$ so we maintain exactness as a corroloary we see that this functor preserves quotients in particular by applying this to the exact sequence

$$0 \longrightarrow N \xhookrightarrow{i} M \xrightarrow{p} M/N \longrightarrow 0$$

1.5 Local Ring, Nakayama's Lemma (the easy proof)

A local ring $(A, \mathfrak{m})$ is a ring with a unique maximal ideal.

Theorem 1.1 (Nakayama's Lemma). *For an ideal $I \subset R$, M finitely generated. If $IM = M$ then there exists $r \in R$, $r = 1 \mod I$ such that $rM = 0$*

In particular in a local ring $mM = M \implies rM = 0$ but since $r = 1 \mod \mathfrak{m}$ it is a unit so $M = 0$. If we replace r with $r - 1$ we get the mneumonic $IM = M \implies im = m$. Additionally it lets us lift generating sets

Lemma 1.2. *If $\tilde{m_1}...\tilde{m_k}$ generate the $A/\mathfrak{m}$ vector space $M/\mathfrak{m}M$ then $m_1...m_k$ generate M*

This just follows from the fact that taking $N = (m_1...m_k)$ then $x \in M, x = x' + \sum a_i m_i = my' + \sum a_i m_i$ so $M/N \subset \mathfrak{m}M/N$ so $\mathfrak{m}M/N = M/N$ so $M/N = 0$ and $M = N$
I dont quite know the "easy" proof. But I know the "hard" one

1.6 The "Determinant Trick" or Cayley-Hamilton theorem, and the proof of Nakayama's lemma via automorphisms

Theorem 1.3 (Cayley-Hamilton). *For for a finitely generated $R-$module M and an ideal I. For any $\phi \in \hom(M, M)$ such that $\phi(M) \subset IM$. There exists a monic polynomial f with other coefficients in I so that $f(\phi) = 0$*

We can then prove Nakayama by taking $\phi = id$ so

$$0 = (id^n + \sum a_i id^i)m = (1 + \sum a_i)m$$

So we let $r = 1 + \sum a_i$ and we're done.
We can also use this to prove that finite $\implies$ integral since we take $\phi_y(m) = ym$. $\phi_y(R) \subset (y)R$ then

$$0 = (y^n + \sum a_i y^i)1$$

So y is integral

1.7 Spec of a ring, its Zariski topology and principal open sets

We define the spectrum of a ring

$$\operatorname{spec} R = \{\mathfrak{p} \subset R | \mathfrak{p} \text{ is a prime ideal } \}$$

With the topology that a subset U is closed if and only if $U = V(I)$ for some ideal I where

$$V(I) = \{\mathfrak{p} \in \operatorname{spec} R | I \subset \mathfrak{p}\}$$

Now for this topology we can define on $X = \operatorname{spec} R$ the principal open sets $X_f = X \setminus V(f)$. These base the topology since

$$U = X \setminus V(I) = X \setminus V((f_\lambda \in I)) = X \setminus \bigcap_\Lambda V(f_\lambda) = \bigcup_\Lambda X_{f_\lambda}$$

And so $\operatorname{spec} R$ is compact since for any open cover this is a union of principal open sets and

$$X = \bigcup_\Lambda X_{f_\lambda} = X \setminus \bigcap_\Lambda V(f_\lambda) = X \setminus V((f_\lambda))$$

So $V((f_\lambda)) = \varnothing \iff (f_\lambda) = R \iff 1 \in (f_\lambda)$ and since 1 must be a finite sum of the f_λ we need only look at this finite set $\{f_i\}$. Then for each f_i we choose some open set containing X_{f_i} then this gives us a finite subcover.

1.8 Noetherian and Artinian conditions on rings and modules

We say that a module is Noetherian if it satisfies the ascending chain condition. That is to say for an ascending chain of submodules

$$I_0 \subset I_1 \subset \ldots$$

There is some n so that $I_m = I_n$ for all $m \geq n$.
Similarly we say that a module is Artinian if for a descending chain of submodules

$$I_0 \supset I_1 \supset \ldots$$

There is some n so that $I_m = I_n$ for all $m \geq n$.

1.9 Finite length modules and Jordan-Hoelder sequences, length optional: proof that length is well defined and additive in s.e.s.

We define the length of a module M to be the largest n so that there is a chain of submodules

$$0 = I_0 \subsetneq \ldots \subsetneq I_n = M$$

If for each k, I_k is a maximal submodule of I_{k+1} we say that this is a composition series. Note that this means that I_{k+1}/I_k is simple so is isomorphic to some $A/\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Ass} I_{k+1}/I_k$. We say that this is the Dévissage of M

For R noetherian and M finitely generated we can guarantee the existence of such a composition series.
To do so we take Σ to be the set of submodules that do have composition series. Note that this is nonempty since 0 has a trivial composition series. Since M is finite over a noetherian ring, M is noetherian so the set Σ has a maximal element say N. If $M \neq N$ then $M/N \neq 0$ so $\text{Ass}(M/N) \neq \varnothing$

THis isn't empty since we take the ideal p that's maximal among $\{\text{ann}\, m | 0 \neq m \in M\}$ which is prime by a standard argument, for $ab \in p,\ a, b \notin p$ we have that for any n $(p + (a))n \neq 0, (p + (b))n \neq 0$ so $(p + (a))(p + (b))n \neq 0$ so $(p + ap + bp + ab)n \neq 0$ which is a contradiction since this is contained in $p = \text{ann}\, m$ so letting $n = m$ we get a problem

So since $\text{ann}\, m = \mathfrak{p} \in \text{Ass}(M/N)$, $N'/N = mM/N \cong A/\mathfrak{p} \subset M/N$. So by definition $N' \in \Sigma$ contradicting maximality of N so $N = M$ and so M has a composition series.

Since in a composition series we choose I_k maximal in I_{k+1} any chain can be increased in length by making it a composition series, squeeze larger ideals between these. So the length of a module is the same as the maximal length of a composition series. And by the Jordan Holder theorem this length is the same for any composition series so is well defined.

Then for a short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

We have that the alternating sum of the lengths is zero, making this look like a very good analogy of dimension. This is clear since a chain for L after inclusion, gives a chain covering the kernel of $M \to N$, then the rest of the module can be covered by the preimage of a chain of N meaning $\ell(M) \geq \ell(N) + \ell(L)$. Additionally a chain on M $M_0 \ldots M_n$ induces a chain on L $M_0 \cap L \ldots M_n \cap L$ and on N $\text{im}(M_0 \to N) \ldots \text{im}(M_n \to N)$. If $L_i = L_{i+1}$ and $N_i = N_{i+1}$ then $M_i = M_{i+1}$ so one of the two must be different so $\ell(M) \leq \ell(N) + \ell(L)$

1.10 optional: proof that Artinian ring is Noetherian, so finite length

For an Artinian ring that isn't Noetherian we take the set of all ideals that arent finitely generated. By the Artinian assumption we take a minimal element of this set I^*

We claim that for any $r \in R$ either $rI^* = I^*$ or $rI^* = 0$. To do so we take the map $\phi_r(x) : I^* \to rI^*$. Then letting K be the kernel $I^*/K \cong rI^*$. If $rI^* = I^*$ we're fine. If not then since $rI^* \subsetneq I^*$ it is finitely generated so K must be infinitely generated so K is an ideal below I^* thats infinitely generated so $K = I^*$ and $rI^* = 0$.

So if we take $r, s \notin \text{ann}(I^*)$ then $rsI^* = sI^* = I$ so $sr \notin \text{ann}(I^*)$ so $p = \text{ann}(I^*)$ is prime so $F = R/p$ is an Artinian domain so a field[1] and I^* is thus a vector space over F so since its not finitely generated over R its not finitely generated over F but any subspace is finitely generated over F since it's finitely generated over R. This is not possible so we have a contradiction so R must've been Artinian

1.11 Associated prime, devissage of a module under Noetherian assumptions

See The Jordan-Hoelder section

1.12 Integral closure inside a finite field extension optional: proof of finiteness

For A, a ring thats integrally closed in it's field of fractions K, then if we have some finite extension L of K, the integral closure B of A in L is finite

$$\begin{array}{ccc} B & \hookrightarrow & L \\ {\scriptstyle finite}\uparrow & & \uparrow{\scriptstyle finite} \\ A & \hookrightarrow & K = A_{(0)} \end{array}$$

1.13 Discrete valuation ring DVR

We say that a ring is a Discrete Valuation ring if it is a principal ideal domain with exactly one nonzero prime (this prime is thus maximal)

1.14 Characterisation of DVRs as 1-dimensional Noetherian local domain that's integrally closed

Clearly a DVR is local, Noetherian and one dimensional by definition since theres only one prime and its a PID.

Then if we have a local, 1D, Noetherian domain write $K = R_{(0)}$ and $\mathfrak{m}$ as the maximal ideal. By Nakayama $\mathfrak{m}^2 \neq \mathfrak{m}$ so we can find some $x \in \mathfrak{m}/\mathfrak{m}^2$. Since

[1] The for $x \neq 0$ the chain (x^n) terminates so $x^n = rx^{n+1}$ so $x^n(rx - 1) = 0$ so $rx = 1$

R has only 2 primes we see that $\mathfrak{m}$ is an associated prime of R/xR so there is some $y \in R$ so that $\mathfrak{m} = \text{ann}(y)$ then we let $a = y/x$. So $\mathfrak{m}a \subset R$ so $a \in \mathfrak{m}^{-1}$. But $a \notin R$ so $R \subsetneq \mathfrak{m}^{-1}$

Note that $\mathfrak{m}\mathfrak{m}^{-1}$ is an ideal of R containing $\mathfrak{m}$. If this is equal to $\mathfrak{m}$ then $a\mathfrak{m} \subset \mathfrak{m}$ so by Cayley-Hamilton a is integral so $a \in R$. This is false so $\mathfrak{m}\mathfrak{m}^{-1} = R$ so $\mathfrak{m}$ is principal since its prime and invertable. Since R is Noetherian and $\mathfrak{m}$ is principal R is a DVR

1.15 optional: Dedekind domain, includes ring of integers of a number field and affine coordinate ring of non-singular algebraic curve

A dedikind domain is a ring such that it is integrally closed, noetherain $1D$ domain. Or as we've just seen this is equivalent to saying that every localisation is a DVR. For example if we take the ring of integers of a number field this is one of these allowing us to factor ideals into prime ideals. This also generalises coordinate rings of nonsingular curves since this is clearly noetherian and $1D$ and since its nonsingular its integrally closed. Therefor looking at Dedikind domains lets us generalise these rings that come up very often.

1.16 Completion, Hensel's lemma, the Artin-Rees Lemma

For an ideal $I \subset R$ we define the $I-$adic completion of a module is the inverse limit

$$\hat{M_I} = \varprojlim M/(I^n M)$$

That is the limit over the natural inverse system $x \to x \mod I^n$. Put more concretely, an element of this ring is a sequence $(a_1, a_2, ...)$ such that $a_i = a_{i+1} \mod I^i$. We say that this is a compatible sequence. Notably this looks like we're saying that a_i, a_{i+1} are in some sense "close" modulo I. So if we say that $v_I(x)$ is the largest n so that $x \in I^n$. Then define $|x| = 2^{-v_I(x)}$ we can consider these as cauchy sequences with respect to this "norm", clearly any cauchy sequence is equivalent to one of these so in some sense its just a topological completion. The one induced by the norm we have here. We say a ring is $I-$adically complete if $R \cong \hat{R}_I$. The main reason we care about such rings is they let us solve equations wayy easier.

Theorem 1.4 (Hensel's Lemma)**.** *Let $(R, \mathfrak{m}, k)$ be a local ring ($k = R/\mathfrak{m}$ the residue field) then if R is $\mathfrak{m}-$adically complete.*
Let $F(x) \in R[x]$ be a monic polynomial, and set $\bar{F} = f \in k[x]$. If f factors as $f = gh$ where g, h are monic and coprime.

Then F has a factorisation $F = GH$ where $G, H \in R[x]$ and

$$\bar{G} = g, \quad \bar{H} = h$$

The proof is done inductively, building up a pair of sequences and then the product must approach our polynomial.
By assumption we can find such a G_1, H_1 that reduce to g, h giving us a factorisation in the first term that is to say $F - G_1H_1 = \sum m_iU_i \in \mathfrak{m}R[x]$. Since g, h are coprime we write $ag + bh = 1$ and so ($u_i = U_i \mod \mathfrak{m}$)

$$gau_i + hbu_i = u_i$$

by adding multiples we can balance this sum such that

$$gv_i + hw_i = u_i \text{ where } v_i = au_i - ch, \quad w_i = bu_i + cg$$

We choose c so that $\deg(v_i) < \deg(h)$. So since u_i, hw_i have degree $< \deg f$ so does hw_i. Choosing lifts $V_i, W_i \in R[x]$ we let

$$G_2 = G_1 + \sum m_iW_i, \quad H_2 = H_1 + \sum m_iV_i$$

Then

$$F - G_2H_2 = F - G_1H_1 - \sum m_i(G_1V_i + H_1W_i) - m_i^2V_iW_i = \sum -m_i^2V_1W_1 \in \mathfrak{m}^2R[x]$$

We then just do this inductively to give a sequence of G_i, H_i such that in each term $F - G_iH_i = 0$ so they are equal

To generalise the construction of this space we define for some directed set of submodules $\{M_\lambda\}$ a topology on M based by the cosets $x + M_\lambda$ in this space it is clear that

1. $\{M_\lambda\}$ base the topology near zero
2. The module operations are continuous
3. If we say M/M_λ is discrete then the quotient maps are continuous
4. The topology is Hausdorff if and only if $\bigcap M_\lambda = 0$

The completion is then the completion with respect to this topology so we take the inverse limit $\hat{M} = \varprojlim M/M_\lambda$. There is a natural inclusion map $a \mapsto (a, a, \ldots)$. This map has kernel $\bigcap_{\lambda \in \Lambda} M_\lambda$ so if this is zero we can treat $M \subset \hat{M}$. We also have natural surjections $M \to M/M_\lambda$, the kernel of which is the completion of the submodule M_λ as a subspace. These themselves induce a topology which is the topology of the completion.

Proposition 1.5. *1. For a short exact sequence of inverse systems*

$$0 \longrightarrow P \longrightarrow Q \longrightarrow R \longrightarrow 0$$

We have an exact sequence

$$0 \longrightarrow \hat{P} \longrightarrow \hat{Q} \longrightarrow \hat{R}$$

2. If also all of the maps in the inverse system $\pi_{i+1} : P_{i+1} \to P_i$ are surjective then the sequence

$$0 \longrightarrow \hat{P} \longrightarrow \hat{Q} \longrightarrow \hat{R} \longrightarrow 0$$

is exact

For the first statement note that the map $c_P : \prod_i P_i \to \prod_i P_i$ defined by $x_i \mapsto \pi_{i+1}(x_{i+1}) - x_i$ has kernel exactly $\hat{P}$ by definition so we just apply the snake lemma to the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \hat{P} & \longrightarrow & \hat{Q} & \longrightarrow & \hat{R} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_i P_i & \longrightarrow & \prod_i Q_i & \longrightarrow & \prod_i R_i & \longrightarrow & 0 \\
 & & \downarrow c_P & & \downarrow c_Q & & \downarrow c_R & & \\
0 & \longrightarrow & \prod_i P_i & \longrightarrow & \prod_i Q_i & \longrightarrow & \prod_i R_i & \longrightarrow & 0
\end{array}$$

Note that the snake lemma gives us a longer sequence,

$$0 \longrightarrow \hat{P} \longrightarrow \hat{Q} \longrightarrow \hat{R} \longrightarrow \mathrm{coker}(c_P)$$

So if we want a real s.e.s. we just need to show that $\mathrm{coker}(c_P)$ is trivial, that is c_P is surjective. If we have that π_i are surjective then we just define $x_1 = 0, \pi_2(x_2) = a_1, \pi_{i+1}(x_{i+1}) = a_i + x_i$, so $c_P(\{x_i\}) = (a_1, a_2 ...)$ so this map is surjective.

We cant yet use this to apply to our completion however since we have sequences of $I^n N, I^n M, I^n(M/N)$ which dont form exact sequences. However in the limit they do if we replace them with some system with the same limit, we do so with the following

Theorem 1.6 (Artin-Rees Lemma). *For A noetherian, I ideal, M finite module, $N \subset M$ then there exists $c > 0$ so that for any $n > c$*

$$I^n M \cap N = I^{n-c}(I^c M \cap N)$$

As a corrolary we also prove that $\hat{N}_I$ is, with respect to the subspace topology, a subspace of $\hat{M}_I$

1.17 Graded ring, Hilbert series, proof that it is a rational function

A graded ring is a ring $R = \oplus_{n \geq 0} R_n$ where each R_n is an abelian group and the multiplication takes $R_a \times R_b \to R_{a+b}$. We tend to restrict to the case that R_0 is artinian or a field. A graded module is the same and satisfies $R_a \times M_b \to M_{a+b}$ and a graded ideal is an ideal so that $I = \sum(I \cap R_n)$.

Note that if R is noetherian this is pretty much just a coordinate ring since R is generated over R_0 by finitely many elements so $R = R_0[x_1 ... x_r]/I$ where we say that $x_i \in R_{d_i}$ has weight d_i, then the weight of any monomial $x^{\{a_i\}}$ is the sum of $a_i d_i$. Additionally since we assume that R_0 is Artinian any finite module has finite length ($\ell(N) < \infty$).

We define the Hilbert series for a module as

$$P(M,t) = \sum_{n=0}^{\infty} P_n(M) t^n \text{ where } P_n(M) = \ell_{R_0}(M_n)$$

This then has the nice consequence that the formal power series is a rational function where $d_i = \text{weight}\, x_i$

$$P(M,t) = \frac{H(M,t)}{\prod_{i=1}^{r}(1 - t^{d_i})}$$

Where $H(M,t) \in \mathbb{Z}[t]$, if we allow M to have negative pieces down to $-s$ we have the same except $H(M,t) \in \mathbb{Z}[t, t^{-1}]$ with degree down to $-s$.

If we're in the nice case where $d_i = 1$ then if H is a polynomial of degree D then for $n \geq D$, $P_n(M)$ is a polynomial in n. In this case after we cancel powers of $1-t$ we can rewrite this as $N(t)/(1-t)^d$ where $N(1) \neq 0$, So $N(t) > 1$ and the order of growth of $P_n(M)$ is

$$N(1) * \frac{n^{d-1}}{(d-1)!} + lot$$

The number d is then the dimension of the graded ring R. This is also the same as one plus the order of the pole at $t = 1$ of $P(R,t)$

2 Second Half

I got so bored of the dimension theory I'm doing homological algebra to make up for it

2.1 General Theorems

We define the ith ext group $\text{Ext}^i(A, B)$ as the ith homology group of the chain $\text{hom}(P_\bullet, B)$ where $P_\bullet$ is a projective resolution of A. Equivalently this is the ith cohomology of the chain $\text{hom}(A, I_\bullet)$ where $I_\bullet$ is an injective resolution of B. Clearly these are the same since we can pass to the dual category to get the same results. Additionally it is an exercise to show that this is the same for any projective resolutions, it simply follows from lifting the maps between the P_i and then this map is a chain homotopy so they induce the same homology. Ext^1 has extra special meaning as it represents the group of extensions of A, B. By this I mean if we have an exact sequence $0 \to A \to X \to B \to 0$ we say this is an extension of A, B, then composition is understood to be the Baer sum where you take the pullback over A of X, Y then quotient this by the inclusion of B.

Theorem 2.1 (Medium Length Exact Sequence in Ext)**.** *For a short exact sequence of $R-$modules and and $R-$module A*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

there exists a connecting homomorphism ∂ such that we have the following exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & \text{hom}(A, L) & \longrightarrow & \text{hom}(A, M) & \longrightarrow & \text{hom}(A, N) & \\ & \swarrow & & & & & \\ & \text{Ext}(A, L) & \longrightarrow & \text{Ext}(A, M) & \longrightarrow & \text{Ext}(A, N) & \end{array}$$

2.2 General definitions

A module is projective if you can lift through surjections ie

$$\begin{array}{ccc} & & N \\ & \overset{\exists}{\nearrow} & \downarrow \\ P & \longrightarrow & M \end{array}$$

We also have the following equivalent characterisations

1. P is projective

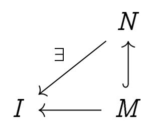
2. There exists Q so that $P \oplus Q$ is free
3. The functor $\hom(P, -)$ is exact
4. For every $R-$module N, $i \geq 1$ $\mathrm{Ext}^i(P, N) = 0$
5. For every $R-$module N, $\mathrm{Ext}^1(P, N) = 0$

We can also conclude that

1. For every finitely presented $R-$module N, $\mathrm{Ext}^1(P, N) = 0$
2. For every finitely generated ideal I of R, $\mathrm{Ext}^1(P, R/I) = 0$

Additionally most of the time projective is equivalent to free, for example if R is a PID or is local

We can also consider the formal dual of this, that is injective modules. Summarised in the following



With alternate characterisations

1. I is injective
2. $\hom(-, I)$ is exact
3. For every $R-$module N, $i \geq 1$ $\mathrm{Ext}^i(N, I) = 0$
4. For every $R-$module N, $\mathrm{Ext}^1(N, I) = 0$
5. For every ideal $J \subset R$, $\mathrm{Ext}^1(R/J, I) = 0$

Where the last is called Baers criterion this can be translated back into the diagram as that we only need check the inclusion $I \hookrightarrow R$ to see injectivity.

2.3 Chapter 5

To do any of this though we really want to actually have resolutions to get our hands on. Free resolutions are really easy to make. You just take the relations and then the relations of the relations and etc. We also have this nice theorem

Theorem 2.2 (Hilbert syzergies + Auslander Buchsbaum)**.** *Suppose M is a finite $S = k[x_1 ... x_n]-$module, then there exists a finite free resolution of the form*

$$0 \longleftarrow M \longleftarrow P_0 \longleftarrow \ldots \longleftarrow P_k \longleftarrow 0$$

With $k \leq n$. If $S, \mathfrak{m}$ is a regular local ring of dimension n, and M is a finite graded $S-$module of $\mathfrak{m}-$depth $\geq d$. Then M has a finite free resolution of length $\leq n - d$

I don't think depth has been defined yet. It can be defined either as the maximal length of a regular sequence in $\mathfrak{m}$ on M, where a regular sequence is a sequence such that each x_{i+1} is regular on $M/(x_1, ..., x_i)M$. Or it can be defined as the smallest i so that $\mathrm{Ext}^i(R/I, M) \neq 0$.

We now have this frankly massive lemma that I might simplify but im just gonna write it out first.

Lemma 2.3. *Let R be a ring, $x \in R$, and let M be a finite $R-$module. Assume that x is a non zerodivisor of M and $xM \subsetneq M$. Additionally either, $(R, \mathfrak{m})$ is local and $x \in \mathfrak{m}$ or A, M, x are graded with $\deg x > 0$. We write $\bar{A} = A/(x), N = M/xM$*

1. ***Generators:** Suppose n_i generate N, then we can find some lift m_i that generate M*
2. ***Relations:** We then define P_0 to be the free $A-$module on this set of generators and $K_0 = \ker(P_0 \to M)$ as the relations. We then do the same to make the free $\bar{A}-$module of generators of N with relations L_0. Then $K_0 \to L_0$ is surjective. That is we can lift each relation on N to a relation on M*
3. ***Syzygies:** A free resolution $Q_\bullet \to N$ can be lifted to a resolution $P_\bullet \to M$ of the same shape. This means that it has the same Betti numbers, and tin the homogeneous case its graded pieces have the same degrees*

We prove this kind of twice. For the first bit the local case is just Nakayama's lemma. Then the graded case we proceed by induction. So for $c \in M$ we write $\pi(c) = \sum a_i n_i$ then pick $b_i \in A$ with $\pi(b_i) = a_i$ then $c - \sum b_i m_i$ is in $\ker \pi$ so is dicisibly by x so $c - \sum b_i m_i = xc'$ where now c' is of smaller degree. So we are now done by induction on the degree.

The meat of the proof then comes from this next bit. We consider the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K_0 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \big\downarrow x & & \big\downarrow x & & \big\downarrow x & & \\
0 & \longrightarrow & K_0 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & L_0 & \longrightarrow & Q_0 & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \big\downarrow & & \big\downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Where all of this is just by construction. Since x is $M-$regular the map $x : M \to M$ is injective. Applying the snake lemma to rows $2, 3$ we get the exact sequence

$$
\begin{array}{ccccccccc}
 & & & & P_0 & \longrightarrow\!\!\!\!\rightarrow & M & & \\
 & & & & \big\| x & & \big\| x & & \\
0 & \longrightarrow & \ker(K_0 \to L_0) & \longrightarrow & \ker(P_0 \to Q_0) & \longrightarrow & \ker(M \to N) & & \\
 & & & & \swarrow & & & & \\
 & & \operatorname{coker}(K_0 \to L_0) & \longrightarrow & \operatorname{coker}(P_0 \to Q_0) & \longrightarrow & \operatorname{coker}(M \to N) & \longrightarrow & 0
\end{array}
$$

Where since the top map $P_0 \to M$ is surjective $\operatorname{coker}(K_0 \to L_0) = 0$ so $K_0 \to L_0$ is surjective

The final part just follows from this by applying it to $P_i \to Q_i$ inductively

This now lets us prove the syzergies theorem by just taking a regular element, then we construct this sequence of decreasing dimension so we end up at zero. If a regular element doesn't exist we just start after step one so we construct the free module of generators and take a regular element there.