Elementary Algebraic Geometry

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Introduction

This is a set of notes I'm making based on the book "Elementary Algebraic Geometry - Klaus Hulek" I do not claim that the presentation here is particularly novel or unique

1 Affine Varieties

The Affine space over k, $\mathbb{A}_k^n$, is basically just k^n, like the information it holds is just that, the new information is the topology you put on it, setting the closed sets to be varieties.

Definition 1.1 (The Zariski Topology) *In the Zariski topology a set is closed if and only if it is the vanishing of some collection of polynomials, that is*

$$X\text{−}closed \iff \exists \{f_i\}_{i\in I} : \ X = \{(x_1, ..., x_n) \in k^n | f_i(x_1, ..., x_n) = 0 \ \forall i \in I\} =: V(\{f_i\}_{i\in I})$$

To immediately make this nicer, if $f(x) = 0$ then $g(x)f(x) = 0$ and if $g(x) = 0$ then $f(x) + g(x) = 0$, that is we can add a bunch of functions to the collection $\{f_i\}_{i\in I}$ without changing the actual set X, we can add enough functions to make the ideal generated by the $f_i's$. This immediately allows us to restrict our attention to $V(I)$ where I is some ideal in $k[x_1...x_n]$. We can also from such a set define an ideal of functions that vanish on this set, call this $I(X)$. This is nice since because k is a field, $k[x_1...x_n]$ is noetherian so the ideal must be finitely generated and so we only need to care about a finite collection of polynomials

This clearly gives a topology as it is easy to check that

Lemma 1.1

1. $V(\langle 0\rangle) = \mathbb{A}_k^n, V(\langle 1\rangle) = \emptyset$
2. $I \subset J \implies V(J) \subset V(I)$
3. $V(I \cap J) = V(I) \cup V(J)$
4. $V(\sum_{\lambda\in\Lambda} J_\lambda) = \bigcap_{\lambda\in\Lambda} V(J_\lambda)$

In a topological space a closed set X is called reducible if $X = X_1 \cup X_2$ where X_1, X_2 are both closed, we can then find that

Lemma 1.2 *Let $X \neq \emptyset$ be an algebraic set with corresponding ideal $I(X)$. Then*

$$X \text{ is irreducible} \iff I(X) \text{ is prime}$$

Since k is a field, $k[x_1 ... x_n]$ is noetherian so $\mathbb{A}_k^n$ is noetherian and using this we can always find some decomposition $X = X_1 \cup ... \cup X_n$ where $X_1 ... X_n$ are irreducible for any closed set X

We can now state the first important theorem of algebraic geometry

Theorem 1.3 (Hilbert's Nullstellensatz) *Let k be an algebraically closed field, and let $A = k[x_1 ... x_n]$. THen t he following hold*

1. *Every maximal ideal $m \subset A$ is of the form*

$$m = \langle x_1 - a_1, ..., x_n - a_n \rangle = I(P)$$

   For some point $P = (a_1 ... a_n) \in \mathbb{A}_k^n$

2. *If $J \subsetneq A$ is a proper ideal, then $V(J) \neq \emptyset$ [This is the namesake of the theorem, the existence of zeroes for any proper ideal]*

3. *For every ideal $J \subset A$ we have*

$$I(V(J)) = \sqrt{J} \quad V(I(J)) = J$$

This gives us the correspondence

$$\begin{tikzcd}
\text{radical ideals} \arrow[r, shift left] & \text{varieties} \arrow[l, shift left] \\
\text{prime ideals} \arrow[u, hook] \arrow[r, shift left] & \text{irreducible varieties} \arrow[u, hook] \arrow[l, shift left] \\
\text{maximal ideals} \arrow[u, hook] \arrow[r, shift left] & \text{points} \arrow[u, hook] \arrow[l, shift left]
\end{tikzcd}$$

We can also define one of the most important objects in algebraic geometry. The coordinate ring

Definition 1.2 (Coordinate Ring) *For an algebraic variety $V \subset \mathbb{A}_k^n$ we define the coordinate ring of V as the quotient*

$$k[V] := k[x_1 ... x_n]/I(V)$$

This is intuitively just all the (polynomial) functions one can define on an algebraic variety. For example take the circle $V(x^2 + y^2 - 1)$ then $k[V]$ will be all functions on V with the equivalence that two are equal if they differ by a multiple of $x^2 + y^2 - 1$ for example $x = x + (x^2 - 1)(x^2 + y^2 - 1)$ since when we evaluate this at any point on the circle, these two functions will give the same output so are the same function.

By looking at these coordinate rings we find that for any of a certain type of algebra can be created as the coordinate ring of some variety so we can find a categorical equivalence

Theorem 1.4 (Categorical equivalence of coordinate rings and varieties) *The functors*

$$F_1 : \textbf{Affine Varieties} \to \textbf{Fintely Generated Reduced } k-\textbf{Algebras}^{op}$$
$$V \mapsto k[V]$$
$$(f : V \to W) \mapsto (F_1 f : k[W] \to k[V], g \mapsto g \circ f)$$

$$F_2 : \textbf{Irreducible Affine Varieties} \to \textbf{Fintely Generated } k-\textbf{Algebras That Are Integral Domains}^{op}$$
$$V \mapsto k[V]$$
$$(f : V \to W) \mapsto (F_2 f : k[W] \to k[V], g \mapsto g \circ f)$$

describe set bijections and more importantly categorical equivalences

Reduced just means that it was formed as the quotient of a radical ideal, alternatively that it has no nonzero nilpotents

This lets us look at the functions that are defined globally on the variety, if however we want more information then it's nice to look at functions that can only be defined somewhere, for example we might want to look at the germs of functions that are defined near a point in order to get information about how the variety looks locally.

Definition 1.3 (Function Field) *The function field of an irreducible variety* V *is* $k(V) := \mathrm{Frac}(k[V])$. *The ring of rational functions defined on* V

We want to look at where these functions are defined so define

Definition 1.4 (Regular) *For* $f \in k(V)$ *and* $P \in V$ *we say that* f *is regular at* P *if there exists* g, h *such that* $f = g/h$ *and* $h(P) \neq 0$

Definition 1.5 (Domain of Definition) *The domain of definition of* f *is then*
$$\mathrm{dom}(f) := \{P \in V | f \textit{ is regular at } P\}$$

In a similar vein to analyse a specific polynomial we can define

Definition 1.6 (Non vanishing of polynomial) *For $h \in k[V]$ we define*

$$V_h := \{P \in V | h(P) \neq 0\} = V - V(h)$$

Is an open set

We can now construct a useful algebraic structure on the variety

Definition 1.7 (Localisation at a point) *The local ring of V at a point P is the ring*

$$\mathcal{O}_{V,P} := \{f \in k(V) | f \text{ is regular at } P\} = k[V]\{h^{-1} | h(P) \neq 0\} = k[V]_{M_P}$$

Where the subscript denotes the localisation (localising at a prime ideal $\mathfrak{p}$ is just taking all fractions that have denominators not in $\mathfrak{p}$) and M_p is the maximal ideal corresponding to the point P, $M_p := \{f \in k[V] | f(P) = 0\} = I(\{P\}) + I(V) \subset k[V]$

Definition 1.8 (Stalk of the Structure Sheaf on a Variety) *For some open subset $U \subset V$ we define*

$$\mathcal{O}(U) := \mathcal{O}_V(U) := \{f \in k[V] | f \text{ is regular on } U\}$$

The collection of all of these along with natural restrictions of corresponding functions gives us a Sheaf on V called the structure sheaf $\mathcal{O}_V$ we can now state a useful theorem

Theorem 1.5 (Domains)

1. $\operatorname{dom}(f)$ *is open and dense*
2. $\mathcal{O}(V) = k[V]$ *that is, the only everywhere regular functions are polynomials*
3. $\mathcal{O}(V_h) = k[V][h^{-1}] =: k[V]_h$ *that is, if f is regular wherever h is nonzero then $f = g/h^m$*

We now have these rational functions but they're more just algebraic objects, how do we see them as functions since theyre not defined everywhere?

Definition 1.9 (Rational Maps)

1. *A rational map $f : V \dashrightarrow \mathbb{A}^n_k$ is a tuple of rational functions $f = (f_1, ..., f_n)$, $f_i \in k(V)$. A map is regular at P if all f_i are regular at P and the domain of definition is* $\operatorname{dom}(f) = \bigcap_{i=1}^n \operatorname{dom}(f_i)$
2. *For an affine variety $W \in \mathbb{A}^n_k$ a rational map $f : V \dashrightarrow W$ is a rational map $f : V \dashrightarrow \mathbb{A}^n_k$ such that $f(\operatorname{dom}(f)) \subset W$*

We want to compose these functions, this may not be possible as for example $x \to (x, 0)$ and $(x, y) \to x/y$ will have a composition that isn't defined anywhere so we want to look at maps such that the composition is always defined somewhere

Definition 1.10 (Dominant) *A rational map $f : V \dashrightarrow W$ is dominant if $f(\operatorname{dom}(f))$ is a dense subset of W*

With this if f is dominant and if g is some rational map the map $g \circ f$ is at least defined on $D = f^{-1}(\mathrm{dom}(g)) \cap \mathrm{dom}(f)$ which cant be empty as $U = \mathrm{dom}(g) \cap f(\mathrm{dom}(f))$ is the intersection of two open dense sets and so is open and dense itself so is a non empty subset of $f(\mathrm{dom}(f))$ so $D = f^{-1}(U)$ is non empty

We since rational maps look like the fractional version of maps between varieties we want to look at the fractional version of maps between coordinate rings. We can easily define

$$Ff : k[W] \to k(V)$$

By taking the same morphism as before, just replacing our variables with the functions, $Ff : g \to g \circ f$ however if $h \in \ker(Ff)$ then $Ff(g/h)$ has no meaning so Ff cannot be a homomorphism $k(W) \to k(V)$ for some $g \in K[W]$ we have

$$g \in \ker(Ff) \iff f(\mathrm{dom}(f)) \subset V(g)$$

But if $f(\mathrm{dom}(f))$ is dense it cannot be contained in a proper closed subset of W (since it then cannot intersect $V - V(g)$ making it not dense) so $V(g) = W$ so $g = 0$ so

$$f \text{ is dominant } \iff Ff : k[W] \to k(V) \text{ is injective}$$

So any dominant map can be extended to a homomorphism $k(W) \to k(V)$

Theorem 1.6 (Rational maps)

1. *A dominant rational map $f : V \dashrightarrow W$ defined a field homomorphism $Ff : k(W) \to k(V)$*
2. *Conversely, a k-homomorphism $\varphi : k(W) \to k(V)$ comes from a uniquely defined dominant rationa map $f : V \dashrightarrow W$*
3. *If f, g are dominant then $F(g \circ f) = Ff \circ fG$*

Theorem 1.7 (Rational maps) *The functor*

F : **Affine Varieties With Dominant Rational Maps** $\to$ **Function Fields**^{op}

Is a categorical equivalence

Definition 1.11 (Quasi-affine variety) *A Quasi-affine variety is an open subset of an affine variety*

We can now construct the category of Quasi-affine varieties

Definition 1.12 (Category of Quasi-affine varieties) *The objects are Quasi-affine varieties and the morphisms $f : V \supset U_1 \to U_2 \subset W$ are rational maps $f : V \dashrightarrow W$ such that $f(U_1) \subseteq U_2$ and $U_1 \subseteq \mathrm{dom}(f)$*

In this category for example, $V_f \cong V$ where $k[V] = k[V_f]$
One last thing we can define is an idea of an abstract affine variety,

Definition 1.13 (Abstract affine variety) *An abstract affine variety over a field k is a pair $(V, k[V])$ consisting of a set V and a k-algebra $k[V]$ of functions on V such that $k[V]$ is generated by finitely many elements $x_1 ... x_n$ over k and the map*

$$V \to \mathbb{A}_k^n$$
$$P \to (x_1(P), ..., x_n(P))$$

Defines a bijection between V and a closed subset of $\mathbb{A}_k^n$

This concludes Chapter 1 (I neglected to include some commutative algebra as well as Noether normalisation as i will bring that up when it comes to defining dimension of a variety)

2 Projective Varieties

Projective space is like normal space but weirder. $\mathbb{P}_k^n$ is essentially the space of lines through the origin.

Definition 2.1 *We define the projective space of dimension n as*

$$\mathbb{P}_k^n := (\mathbb{A}_k^{n+1} - \{0\})/\sim$$

Where $\sim$ is the equivalence relation given by

$$x \sim y \iff kx = ky$$

Note that k is the base field so kx refers to the span of x

To visualise this we can imagine taking some n-dimensional hyperplane in $\mathbb{A}_k^{n+1}$, say $\{x_1 = 1\}$ then this will intersect "most" of the lines through the origin, each point on this hyperplane corresponds to a unique element of $\mathbb{P}_k^n$, we just miss all of the lines that lie parallel to the hyperplane but since the hyperplane is just a copy of $\mathbb{A}_k^n$, so is the plane perpendicular, so the lines we miss will just be all of the lines that lie in $\mathbb{A}_k^n$ so a copy of $\mathbb{P}_k^{n-1}$

$$\mathbb{P}_k^n = \mathbb{A}_k^n \sqcup \mathbb{P}_k^{n-1}$$

For any point $x \in \mathbb{P}_k^n$ we write $x = (x_0 : x_1 : ... : x_n)$ where x is the image of $(x_0, x_1, ..., x_n)$ under the quotient map. Note that since 0 is excluded from $\mathbb{P}_k^n$ we can cover $\mathbb{P}_k^n$ by ensuring each of the coordinates is non zero

$$\mathbb{P}_k^n = U_0 \cup U_1 \cup ... \cup U_n$$

Where each $U_i = \{(x_0 : ... : x_n) \in \mathbb{P}_k^n | x_i \neq 0\}$ is isomorphic to $\mathbb{A}_k^n$ since we can scale such that $x_i = 1$ and then the rest of the variables are free.

Now we have an idea of the geometry of $\mathbb{P}_k^n$ we better find some geometric objects. The issue is that, for example, if we try to take $C = V(x^2 - y)$ in $\mathbb{P}_k^n 1$ (where $\mathrm{char}(k) \neq 2$) then $(1 : 1) \in C$ since $1^2 - 1 = 0$ but $(1 : 1) = (2 : 2) \notin C$ since $2^2 - 2 = 2 \neq 0$. To remedy this we look at what are called homogeneous polynomials, where the degree of each term is the same.

Definition 2.2 (Homogeneous polynomial) *We say that $f \in k[x_0...x_m]$ is homogeneous of degree n if*

$$f = \sum_{i=0}^{\ell} a_i \mathbf{x}^{\mathbf{u_i}}$$

Where we use multi degree notation, and for every i, $\mathbf{u_i} \in \mathbb{N}^{m+1}$ its sum of entries is $=: |\mathbf{u_i}| = n$

This solves our issue as if $f(\mathbf{x}) = 0$ then $f(\lambda\mathbf{x}) = \sum_{i=0}^{\ell} a_i(\lambda\mathbf{x})^{\mathbf{u_i}} = \lambda^n \sum_{i=0}^{\ell} a_i \mathbf{x}^{\mathbf{u_i}} = \lambda^n f(\mathbf{x}) = 0$

This may feel like an annoying restriction but we lost no information from the $\mathbb{A}_k^n$ case since for any polynomial

$$k[x_1 ... x_n] \ni f = \sum_{i=0}^{\ell} a_i \mathbf{x}^{\mathbf{u_i}}$$

We can homogenise it by adding a variable to each of the terms to make the degrees match, letting $v = \max |\mathbf{u_i}|$ we can construct the homogeneous polynomial

$$k[x_0, x_1 ... x_n] \ni f = \sum_{i=0}^{\ell} a_i x_0^{v-|\mathbf{u_i}|} \mathbf{x}^{\mathbf{u_i}}$$

Where when we look at the restriction to just U_0 we get back our original polynomial with a corresponding variety in $\mathbb{A}_k^n$. Before we define varieties in $\mathbb{P}_k^n$ however we don't just want to look at individual polynomials, we want to look at ideals.

Clearly we cant just look at the zero locus of an ideal generated by homogeneous polynomials since $x^2 - y \in \langle x^2, y \rangle$ and we've already concluded that that polynomial is a pain when we're in projective space. So we first take some generality,

Definition 2.3 (Graded Ring) *R is a graded ring if $R = \bigoplus_{d \geq 0} R_d$ where each R_d is an abelian group and $R_a \cdot R_b \subseteq R_{a+b}$, $R_c \cap R_d = \{0\}$ for $c \neq d$, an element of R_d is called a homogeneous element of degree d*

The important example we care about here is the polynomial ring, which is a direct sum of the groups of polynomials of degree d

$$k[x_0, ..., x_n] = \bigoplus_{d \geq 0} k_d[x_0 ... x_n]$$

Definition 2.4 (Homogeneous Ideal) *A homogeneous ideal is then an ideal of R that can be written as*

$$I = \bigoplus_{d \geq 0} (I \cap R_d)$$

That is, I is generated by its homogeneous elements.

We can now construct the projective variety.

Definition 2.5 *If $I \subset k[x_0 ... x_n]$ is a homogeneous ideal. The projective variety of I is*

$$\mathcal{V}(I) := \{(y_0 : y_1 : ... : y_n) \in \mathbb{P}_k^n | f(y_0, y_1, ..., y_n) = 0 \ \forall f \in I \cap k_d[x_0 ... x_n] \ \forall d\}$$

That is, for any homogeneous polynomial in the ideal, it evaluates to zero at this point

Much of the standard algebra holds over these varieties, for example by defining

Definition 2.6 (Projective ideal of a set) *The homogenous ideal of a set $X \subset \mathbb{P}_k^n$ is the ideal*

$$\mathcal{I}(X) = \langle f \in k_d[x_0...x_n] \mid f(y_0, ..., y_n) = 0 \ \forall (y_0 : ... : y_n) \in X, \ d \in \mathbb{N} \rangle$$

And using that $\mathbb{P}_k^n$ is a quotiented version of $\mathbb{A}_k^{n+1}$ we can turn it back giving

Definition 2.7 *For a homogenous ideal I, $V(I) = \pi^{-1}(\mathcal{V}(I)) \cup \{0\}$ is called the projective cone of $\mathcal{V}(I)$*

Allowing us to apply affine results to the projective case.

Theorem 2.1 (Projective Nullstellensatz) *Let k be an algebraically closed field. Then for a homogeneous ideal J we have*

1. $\mathcal{V}(J) = \emptyset \iff \sqrt{J} \supset \langle x_0...x_n \rangle$ *This is called "The Irrelevant Ideal"*
2. *If $\mathcal{V}(J) \neq 0$ then $\mathcal{I}(\mathcal{V}(J)) = \sqrt{J}$*

Giving the correspondance Finally we can justify that we lose nothing when

$$\begin{tikzcd}
\text{Proper Hom-Radical Ideals} \arrow[r, leftrightarrows] & \text{Projective Varieties} \\
\text{Proper Hom-Prime Ideals} \arrow[r, leftrightarrows] \arrow[u, hook] & \text{Irred-Projective Varieties} \arrow[u, hook]
\end{tikzcd}$$

going to the projective case as going back to the covering $\mathbb{P}_k^n = U_0 \cup ... \cup U_n$

Theorem 2.2 (The cover really is affine) *The map*

$$\begin{aligned} j_i : U_i &\to \mathbb{A}_k^n \\ (x_0 : ... : x_i : ... : x_n) &\mapsto (x_0/x_i, ..., x_{i-1}/x_i, x_{i+1}/x_i, ..., x_n/x_i) \end{aligned}$$

Is a homeomorphism

So any information about the varieties in $\mathbb{A}_k^n$ can be gleamed from its homogenised version in $\mathbb{P}_k^n$ by just looking at the restriction to U_0. Additionally by considering the covering $\mathbb{P}_k^n = U_0 \sqcup \mathbb{P}_k^{n-1}$ we have

Theorem 2.3 (Bijection of projective and affine) *The map $\mathbb{P}_k^n \supset X \mapsto j_0(X \cap U_0) \subset \mathbb{A}_k^n$ gives a bijection*

$$\{\textit{Irreducible projective varieties } X \textit{ with } X \not\subset \{x_0 = 0\}\} \longleftrightarrow \{\textit{Irreducible affine varieties}\}$$

We can now talk about the second part of the theory of affine varieties, rational maps and function fields. This isn't immediately clear as for some homogeneous polynomials f, g if they have differing degrees then $f(\lambda x)/g(\lambda x) = \lambda^{m-n} f(x)/g(x)$ so f/g is only a function on $\mathbb{P}_k^n$ if the degrees of f and g match

Definition 2.8 *The function field of a projective variety V is defined as*

$$k(V) = \left\{ \frac{f}{g} \mid f, g \in k_d[x_0 ... x_n] \ d \in \mathbb{N} \right\}$$

where $f/g = f'/g'$ iff $fg' - gf' \in \mathcal{I}(V)$

Lemma 2.4 *If we do the standard cover of $\mathbb{P}_k^n = U_0 \cup ... \cup U_n$ we can cover a variety $V = V_0 \cup ... \cup V_n$ then,*

$$k(V) \cong k(V_0)$$

Thanks to our work this really does represent the ring of functions defined on (some open subset of) the variety, we would also like a coordinate ring that dispite not being functions still affords use in theory

Definition 2.9 (Homogeneous coordinate ring) *For some projective varitey V with affine cone V^α we define the homogeneous coordinate ring of V*

$$S(V) := k[V^\alpha] := k[x_0, ..., x_n]/\mathcal{I}(V)$$

Which is a graded ring since

$$S(v) = \bigoplus_{d \geq 0} S_d(V)$$

Where

$$S_d(V) := \{f \in S(V) | f \text{ is homogeneous with } \deg f = d\} \cup \{0\}$$

Additionally

$$k(V) \cong S(V)_{(\langle 0 \rangle)}$$

Where we will define this notion now

If we want to take a localisation of a graded ring without losing the graded aspect, first note that for f, g homogeneous polynomials the degree of $f/g \in S_T$ (which is clearly just $\deg f - \deg g$) is well defined since for $f'/g' = f/g$ we have that for some h $h(f'g - fg') = 0$ and so $hf'g = hfg'$ so $\deg h + \deg f' + \deg g = \deg h + \deg f + \deg g$ so $\deg f - \deg g = \deg f' - \deg g'$

Definition 2.10 (Localisation of Graded ring) *For a multiplicatively closed system T*

$$R_{(T)} := \left\{ \frac{f}{g} \in R_T | \frac{f}{g} \text{ is homogeneous of degree } 0 \right\}$$

For a prime ideal $\mathfrak{p}$ we define $T_\mathfrak{p} := \{f \in R | f \text{ is homogeneous, } f \notin \mathfrak{p}\}$

$$R_{(\mathfrak{p})} := R_{(T_\mathfrak{p})}$$

If R is an integral domain then for $f \in R$ we define $T_f = \{f^n | n \in \mathbb{N}\}$

$$R_{(f)} := R_{(T_f)}$$

To finish these kinds of constructions we consinder regularity for functions

Definition 2.11 (All the boring bits) *f is regular at P if there are some g, h such that $f = g/h$ $h(P) \neq 0$ (h isn't a function on V but it equaling zero is okay since if one representation of a point evaluates to zero, all representations do)*

$\operatorname{dom} f$ is defined as all the points P such that f is regular at P

M_P is defined as $\{f \in S(V) | f \text{ homogeneous } f(P) = 0\}$

$\mathcal{O}_{V,P} := \{f \in k(V) | f \text{ is regular at } P\} \cong S(V)_{M_P}$

This local ring has corresponding maximal ideal $m_{V,P} := \{f \in \mathcal{O}_{V,P} | f(P) = 0\}$

For a quasi projective variety $U \subset V$ the ring of regular functions on U is defined by $\mathcal{O}(U) := \{f \in k(V) | U \subset \operatorname{dom} f\} = \bigcap_{P \in U} \mathcal{O}_{V,P}$ Giving us some sort of structure sheaf on a variety.

The following theorem is almost cool, until you realise its just saying that if f/g is regular everywhere then g kinda has to be a constant function and the only way for this to have degree zero is if f is also constant

Theorem 2.5 *If V is an irreducible projective variety defined over an algebraically closed field k, then every regular function on V is constant, $\mathcal{O}(V) \cong k$*

The proof requires some module theory that i assume you already know.
Im now quickly going to redefine all the obvious statements about rational maps, gimme a sec

Definition 2.12 (Rational maps) *A rational map $f : V \dashrightarrow \mathbb{P}^n_k$ is a tuple $(f_0 : ... : f_n)$ of rational functions $f_i \in k(V)$*

A rational map is regular at P if each f_i is regular at P and for at least one i $f_i(P) \neq 0$

We can have rational maps between varieties given by $f : V \dashrightarrow W$ if $f : V \dashrightarrow \mathbb{P}^n_k$ and $f(\operatorname{dom} f) \subset W$

For quazi projctive or affine varieties we have morphisms $f : U_1 \to U_2$ if $f : V \supset U_1 \dashrightarrow U_2 \subset W$ where $U_1 \subset \operatorname{dom} f$ and $f(\operatorname{dom} f) \subset U_2$

For example the homeomorphism we defined earlier is an isomorphism of quazi varieties

Definition 2.13 *(V, W irreducable quazi varieties) We say that $f : V \dashrightarrow W$ is a birational equivalence is there is some rational map $g : W \dashrightarrow V$ such that*

$f \circ g = id_W, g \circ f = id_V$ *where this equality is understood to mean equality on an open dense subset as these maps may not be defined everywhere. We then say that* V, W *are birationally equivalent*

Theorem 2.6 *The following are equivalent*

1. f *is birational*
2. f *is dominant and* $Ff : k(W) \to k(V)$ *is an isomorphism*
3. *There are open sets* $V_0 \subset V, W_0 \subset W$ *such that the restriction* $f|_{V_0} : V_0 \to W_0$ *is an isomorphism*

In addition to this we have that

Theorem 2.7 *For* L/K *a finitely generated field extension which is not finite, where* $\operatorname{char} K = 0$*. Then we may find a quasi projective variety* V *such that the field extension* $K(V)/K \cong L/K$

Theorem 2.8 *There is a contrapositive equivalence of categories*

$F : \textbf{Irred Quasi Proj Varieties w rational maps} \to \textbf{f} - \textbf{g field extentions of } k^{op}$