Vector Bundles

Kyle Thompson

May 2023

Contents

1 Sheaves

If we aspire for full generality we would want to talk about Grothendieck topologies and presheaves, luckily we don't, we can just do the easy version.

Definition 1.1 (Presheaf of Rings)**.** If we have a topological space X we let σ_X denote its poset of open sets. Then a presheaf of rings is a functor $\mathcal{O} : \sigma_X^{op} \to$ **Ring** where we call the image of the morphism $U \subset V$, $\text{res}_{V,U}$.

For such a presheaf to be called a sheaf it must satisfy the additional gluing criterion

Definition 1.2 (Sheaf of rings)**.** A presheaf $\mathcal{O}$ is a sheaf if for any $f|_U \in \mathcal{O}(U), f|_V \in \mathcal{O}(V)$ such that $\text{res}_{U,U\cap V}(f|_U) = \text{res}_{V,U\cap V}(f|_V)$ there is a unique $f \in \mathcal{O}(U \cup V)$ such that $\text{res}_{U\cup V,U} = f|_U$ and $\text{res}_{U\cup V,V} = f|_V$

The prototypical example of a sheaf is that of a collection of functions (often we just consider sheaves of sets but as far as I care it's always sheaves of rings). For example one might want to look at a space by considering the set of functions from this space to one we know better. For example in differential geometry the natural sheaf to consider for a manifold is all of the smooth functions $M \to \mathbb{R}$ where the induced maps really are just restrictions

$$\mathcal{O}_M(U) = \{f : U \to \mathbb{R} : f \text{ is smooth}\}$$

Note that since $\mathbb{R}$ is just a lovely lovely space this is in fact a sheaf of rings since there's a natural way to add, subtract and multiply functions to $\mathbb{R}$. In fact this sheaf is so nice one may use it as the definition of a manifold. Although to do that definition justice we need to do a little more work.

When looking at manifolds we often want to find out properties at certain points, tangent spaces for example. To do so we want to know what functions there are "at a point". We can all agree that the functions $x \to x, x \to x^2$ are demonstrably different, even if you're stuck at the point 0 since looking to your sides you see that these functions diverge away from each other. If we want to look at the derivative for example we really only care about what 0 "sees". In differential geometry we would call these Germs of functions. We generalise this with the notion of a stalk.

Definition 1.3 (Stalk at a point)**.** For a sheaf $\mathcal{O}_X$ we define the Stalk at $x \in X$, $\mathcal{O}_{X,x} = \mathcal{O}_x$ as the set

$$\mathcal{O}_x = \bigsqcup_{U \ni x} \mathcal{O}(U)/\sim$$

Where $f \in U \sim g \in V$ if there is some $N \subset U \cap V$ such that $\text{res}_{U,N}(f) = \text{res}_{V,N}(f)$. The ring operations are then defined on the largest neighborhood where the two objects are defined

This definition is a bit of a set theoretic mess, what we're really describing though is just the colimit over the full subcategory of neighborhoods of x.

$$\mathcal{O}_x = \varinjlim_{U \ni x} O(U)$$

(Since this happens to be a directed poset it somewhat more descriptive to use a direct limit instead of the more general colimit)

Definition 1.4 (Ringed Space)**.** A ringed space is a pair $(X, \mathcal{O}_X)$ of a space and a sheaf on the space

Now the following definition I kinda hate. Since this ringed space is assigning a ring to each neighborhood of the space it seems like it is already a locally ringed space. However the word locally ringed really means (locally ring)-ed.

Definition 1.5 (Locally ringed space)**.** A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ such that at each point, the stalk $\mathcal{O}_x$ is a local ring. That is to say it has a unique maximal ideal $\mathfrak{m}_x$

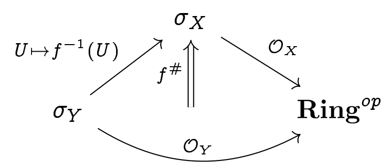
I'm realising this is a bit of a diversion from the point of this document but I just want to get this all down. I promise we'll get to that definition of a manifold soon. To do so we do however need to develop the correct categories for these ringed spaces.

Definition 1.6 (Morphism of Ringed Spaces)**.** For ringed spaces $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ a morphism of ringed spaces $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair of maps $f, f^{\#}$ where f is a continuous map $f : X \to Y$ and $f^{\#}$ is a map of sheaves $f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$. Note the backwards ordering and the f_*. This is essentially just a consequence of the maps needing to type-check

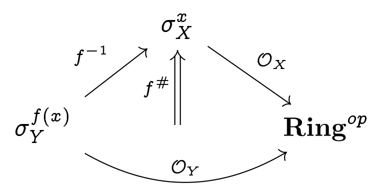
Definition 1.7 (Morphism of Locally Ringed Spaces)**.** For locally ringed spaces $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ a morphism of locally ringed spaces $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that the induced map of stalks $f_x^{\#} : \mathcal{O}_{f(x)} \to \mathcal{O}_x$ is a local ring map, that is to say $f_x^{\#}(\mathfrak{m}_{f(x)}) \subset \mathfrak{m}_x$

This was kinda pointless but useful since for now we're just going to be talking about isomorphisms and any isomorphism is automatically a local ring map. We're almost there don't worry

For future reference I think its worthwhile to spend some time explaining the induced map a bit better we can define it by doing the obvious thing to the representatives of equivalence classes but since it comes up a lot, it's worth diving into the abstract nonsense. If we have a map of ringed spaces $f : X \to Y$ we have a family of maps $f^{\#}(U) : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ that agree with the restriction maps. This means that these maps form a natural transformation



So we say that $f^\#$ is a natural transformation $\mathcal{O}_Y \to \mathcal{O}_X \circ f^{-1}$. By restricting σ_Y to the smaller category $\sigma_Y^{f(x)}$ of open sets over $f(x)$. For any open set U over $f(x)$, $f^{-1}(U)$ will be an open set over x so we can restrict σ_x to the category σ_X^x giving the diagram



We then define the stalk $\mathcal{O}_{Y,f(x)}$ as the colimit of this functor $\mathcal{O}_Y$. That is we have a natural transformation $res_{f(x)} : \mathcal{O}_Y \to \Delta\mathcal{O}_{Y,f(x)}$ such that any natural transformation $\mathcal{O}_Y \to \Delta a$ factors through $res_{f(x)}$. We also define $\mathcal{O}_{X,x}$ as the colimit of this $\mathcal{O}_X$. By precomposing with f^{-1}, in the diagram category $[\sigma^{op}_{Y,f(x)}, \mathbf{Ring}]$ we have the following diagram

$$\begin{array}{ccc} \mathcal{O}_Y & \xrightarrow{f^\#} & \mathcal{O}_X \circ f^{-1} \\ \downarrow{\scriptstyle res_{f(x)}} & & \downarrow{\scriptstyle res_x|_{\mathcal{O}_X\left(f^{-1}\left(\sigma_Y^{f(x)}\right)\right)}} \\ \Delta\mathcal{O}_{Y,f(x)} & & \Delta\mathcal{O}_{X,x} \end{array}$$

Where we've restricted the res_x to the image of this new functor, since res_x refers to a family of maps in **Ring** restricting to a smaller image still gives us a natural transformation. Since the map $res_{f(x)}$ is the universal cone and we have another cone $res_x|_{\mathcal{O}_X(f^{-1}(\sigma_Y^{f(x)}))} \circ f^\#$ this map factors uniquely as $g \circ res_{f(x)}$ so we define the map of stalks to be $g(Y) : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$

Definition 1.8 (Local isomorphism)**.** We say that a (locally) ringed space $(X, \mathcal{O}_X)$ is locally isomorphic to some other (locally) ringed space $(Y, \mathcal{O}_Y)$ if for every point $x \in X$ there is an open neighborhood U of x such that there is an isomorphism of ringed spaces, $(U, \mathcal{O}_X|_U) \to (Y, \mathcal{O}_Y)$

Ok we're here

Definition 1.9 (Manifold)**.** An $n-$dimensional smooth manifold is a locally ringed space $(M, \mathcal{O}_M)$ that's locally isomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ where $\mathcal{O}_{\mathbb{R}^n}$ is the natural sheaf of $\mathcal{C}^\infty$ functions $\mathbb{R}^n \to \mathbb{R}$

The other main object studied in Geometry is the scheme, we start with the easy ones

Definition 1.10 (Affine Scheme)**.** An Affine Scheme is the locally ringed space $(\operatorname{spec} A, \mathcal{O}_{\operatorname{spec} A})$ where $\operatorname{spec} A$ is the spectrum of A with its Zariski topology and $\mathcal{O}_{\operatorname{spec} A}$ assigns to an open set

$$\mathcal{O}_{\operatorname{spec} A}(U) = \left\{ s : \operatorname{spec} A \to \bigsqcup_{\mathfrak{p} \in \operatorname{spec} A} A_{\mathfrak{p}} : s(\mathfrak{p}) \in A_{\mathfrak{p}} \text{ and LAF} \right\}$$

LAF : $\forall \mathfrak{p}$,there is an open neighborhood U where, for fixed a, b, $s(\mathfrak{q}) = \frac{a}{b}$for any $\mathfrak{q}$ in this neigborhood

This is essentially designed so that $\mathcal{O}_{\mathfrak{p}} = A_{\mathfrak{p}}$, and later we will define module sheaves such that $\tilde{\mathcal{M}}_{\mathfrak{p}} = M_{\mathfrak{p}}$. In fact, the definition here that looks a little odd is exactly the definition we get by running $\mathcal{O}_p = A_p, \mathcal{O}(X_f) = A_f$ through the construction we will later do on modules, so it's not just plucked out of the sky

Definition 1.11 (Scheme)**.** A scheme is a locally ringed space, locally isomorphic to an Affine Scheme

Once we have a locally ringed space, it seems silly to try to put more rings on this so when we study these spaces we adjoin some other sheaf to it, the most common is a thing called a sheaf of $\mathcal{O}_X-$modules

Definition 1.12 (Sheaf of $\mathcal{O}_X-$modules)**.** For a ringed space $(X, \mathcal{O}_X)$ a sheaf of $\mathcal{O}_X-$modules is a sheaf $\mathcal{F}$ of abelian groups such that for every open set U, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)-$module where the restriction maps are also module homomorphisms. We then say we have a morphism of sheaves of $\mathcal{O}_X-$modules if we have morphism of sheaves of abelian groups where all group homomorphisms are also module homomorphisms

2 Modules as Sheaves

If we have some affine scheme $X = \operatorname{spec} A$ we have the Zariski topology on it, that is the topology based by X_f where $X_f = V(f)^c$ are called principal open sets. In the case of polynomial rings this corresponds to $X_f = \{f \neq 0\}$.

Recalling from before what we want for a sheaf associated to a module, calling the (pre)sheaf associated to M, $\mathcal{M}$ we wanted $\mathcal{M}_p \cong M_p$ but $M_p \cong M \otimes A_p$ and $M \otimes A_p \cong M \otimes \mathcal{O}_p$ so a reasonable thing to try would be to define $\mathcal{M}(U) = M \otimes \mathcal{O}_X(U)$ however this has a slight problem, it's not a sheaf. To make it a sheaf we want to do something called the sheafification of a presheaf and to find this object with such a wonderful name we're going to take the scenic route. Sheaves are designed as algebraic structures that capture the personality of "functions to a space" so naturally if we have a set of functions to a space it's going to form a sheaf.

So what we want to do is make a space so that the functions on U look roughly like $M \otimes \mathcal{O}_X(U)$. To do this we construct what is called the espace etale for a sheaf, in fact initially sheaves were defined as just spaces over another where the projection was locally a homomorphism, then one would look at the sections to get what we now call the sheaf. So lets start, we have our scheme X or better yet, we have our locally ringed space $(X, \mathcal{O}_X)$ where $\mathcal{O}_X(X) = A$ then we want to find a space over X, i.e. a space E and a projection map $\pi : E \to X$. Looking at this space we see that the stalks are a lot like just the preimages $\pi^{-1}(x)$ so[1] we want the space to contain all of these and they then map down onto x, so as a set we define, calling our presheaf $\mathcal{F}$ for now

$$\operatorname{Spe}(\mathcal{M}) = \bigsqcup_{x \in X} \mathcal{F}_x$$

We now want the sections of this space to be the sections of $\mathcal{F}$, to do this we can kind of cheat, now we have the space we can turn each $s \in \mathcal{F}(U)$ into a section on the space by mapping $x \to s_x$ and then taking the strongest topology such that these are our sections. Now taking $\tilde{\mathcal{F}}(U)$ to be the module[2] $\{s : X \to \operatorname{Spe}(\mathcal{F}) | s \text{ is a continuous section}\}$. This clearly maintains the local propertys of the presheaf it just adds in the correct maps to allow for gluing

Definition 2.1 (Sheafification). For a presheaf $\mathcal{F}$ on X we define the sheafification of $\mathcal{F}$ (also called the sheaf associated to $\mathcal{F}$) as the sheaf

$$\tilde{\mathcal{F}}(U) = \{s : X \to \operatorname{Spe}(\mathcal{F}) | s \text{ is a continuous section}\}$$

[1] in the case of a discrete setup all sections are locally constant so the elements of the preimage are exactly the stalk

[2] since the $s(x) \in \mathcal{F}_x$ is a module we can add and scalar multiply the functions making the set of sections a module

This leads to the construction

Definition 2.2 (Sheaf associated to a Module)**.** Given a locally ringed space $(X, \mathcal{O}_X)$. Letting $A = \mathcal{O}_X(X)$ and taking an $A-$module M we define the sheaf of $\mathcal{O}_X$ modules associated to M as

$$\tilde{\mathcal{M}}(U) = \left\{ s : \operatorname{spec} A \to \bigsqcup_{\mathfrak{p} \in \operatorname{spec} A} M_{\mathfrak{p}} : s(\mathfrak{p}) \in M_{\mathfrak{p}} \text{ and LAF} \right\}$$

With the natural restriction maps

Note that $\tilde{\mathcal{M}}(X_f) = M_f$, we could've instead taken this as a starting point but I feel that this construction is more general

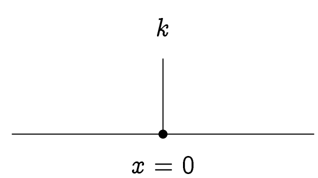
As an example if we consider the ring $A = k[x]$ we can look at a simpler case since $\operatorname{spec} A \sim \mathbb{A}^1_k$. If we then consider some toy module such as $M = k[x]/(x) \cong k$ we can see that if we have some open set X_f not containing zero then $f(0) = 0$ so

$$\tilde{\mathcal{M}}(X_f) = M_f = M[f^{-1}] = M[0^{-1}] = 0$$

However if the set does contain zero then suddenly we get some stuff since $f(0) = a \neq 0$

$$\tilde{\mathcal{M}}(X_f) = M_f = M[f^{-1}] = M[a^{-1}] = M = k$$

This is called a skyscraper sheaf since it jumps suddenly at one point and is zero everywhere



3 Vector Bundles

For a manifold we have certain ways to make new manifolds from old, one way of doing so, where we can make shapes such as the Möbuius band, is by constructing vector bundles. Essentially we can attach a vector space to each point in the manifold in a way such that it's locally very well behaved (locally its just a product we say this is is the property of being locally trivial). Taking the standard definition for manifolds based on charts we define,

Definition 3.1 (Vector Bundle)**.** A vector bundle of rank r on a smooth manifold M is a smooth manifold E with a smooth map

$$\pi : E \to M$$

Such that there exists an open cover $\{U_i\}$ of M where (U_j, φ_j) are charts with the property that

1. There is a diffeomorphism f_j such that the following commutes

$$\begin{array}{ccc} \pi^{-1}(U_j) & \xrightarrow{f_j} & U_j \times \mathbb{R}^r \\ & \searrow^{\pi} & \downarrow{\scriptstyle \pi_j} \\ & & U_j \end{array}$$

2. For $p \in U_j \cap U_k$, If $(p, x) \in U_j \times \mathbb{R}^r$, $(p, y) \in U_k \times \mathbb{R}^r$ then

$$f_j \circ f_k^{-1}(p, y) = (p, f_{jk}(p) \cdot x)$$

   Where $f_{jk} : U_j \cap U_k \to \mathrm{GL}(r, \mathbb{R})$ are smooth

Given such a vector bundle we can construct a sheaf made up of the sections of this bundle

Definition 3.2 (Section Sheaf)**.** Given a vector bundle (E, π) on M we define the section sheaf

$$\mathcal{V}(U) = \{f : U \to E : \pi \circ f(p) = p \ \forall p \in U\}$$

Where the restrictions are obvious

By pre-composition we see that this is a sheaf of $\mathcal{O}_M-$modules.

Looking at the example of the trivial bundle we take $M \times \mathbb{R}^r$ then a section f looks like $u \to (u, f_1(u), ..., f_r(u))$ where each $f_i : U \to \mathbb{R}$, that that is to say there is a natural identification between sections of this bundle and $r-$tuples of elements of $\mathcal{O}_M(U)$, so $\mathcal{V}(U) = \mathcal{O}_M(U)^r$ woah, thats a nice sheaf, in fact, any vector bundle will be almost as nice! Since vector bundles are locally trivial locally this will always be true. We say that this sheaf is locally free of rank r.

4 Vector Bundles as Sheaves

What just happened was we found out that every vector bundle of rank r corresponds to a locally free sheaf of rank r. Of course we're only looking at manifolds right now but that's fine. It turns out that in any case where the objects we have have any kind of meaning we see that this is still the case. In fact the correspondence is so nice that we might aswell take it as a definition

Definition 4.1 (Vector Bundles II)**.** A vector bundle of rank r on a ringed space $(X, \mathcal{O}_X)$ is a locally free sheaf of $\mathcal{O}_X-$module of rank r

Since if we have a vector bundle, we get a locally free finite rank sheaf. If instead we start with a locally free sheaf, we want to make a vector bundle, assuming we're working over manifolds again[3] we take the cover $\{U_i\}$ where $\mathcal{F}(U_i) \cong \mathcal{O}_M(U)^r$ then we take the space made up of $U_i \times \mathbb{R}^r$ and we just need to find out how to glue them together, since we're on a sheaf gluing is easy. Taking our isomorphisms $f_i : \mathcal{F}(U_i) \to \mathcal{O}_M(U)^r$ we restrict to a map $f_{ij} = f_j|_{U_i \cap U_j} \circ f_i|_{U_i \cap U_j}^{-1}$ where $f_{ij} : \mathcal{O}_M(U_i \cap U_j)^r \to \mathcal{O}_M(U_i \cap U_j)^r$ which is just some linear map defined on the basis elements so a matrix of functions $U_i \cap U_j \to \mathbb{R}$. So we take our $U_i \times \mathbb{R}^r$, $U_j \times \mathbb{R}^r$ and glue them along the map $(p, x) \mapsto (p, f_{ij}(p) \cdot x)$ these maps give us the cocycles seen in (2.) in the definition of a vector bundle over a smooth manifold

[3] since for a general space we don't have a notion of what it means to be a vector bundle this is all we can really do, there is a notion for varieties but manifolds are more intuitive and the construction is identical

5 Modules as Vector bundles

This gives us a lovely correspondence between geometry and algebra, since if we have sufficiently nice modules we can treat them as vector bundles and we can treat vector bundles entirely sheaf theoretically. For example any finitely generated projective module will induce a locally free sheaf so we can study projectives via looking at vector bundles. One of the most important objects in geometry as a whole is the line bundle, that is a vector bundle of rank 1. A fun example of such an object is that fractional ideals of a number field K are just the line bundles over $\operatorname{spec} \mathcal{O}_K$

Theorem 5.1 (Serre-Swan Correspondence). *For a commutative Noetherian ring A, passing a module to its sheaf induces an equivalence of the category of finitely generated projective $A-$modules and the category of algebraic vector bundles over A, that being locally free sheaves of $\mathcal{O}_{\operatorname{spec} A}$ modules*

This in fact also applies for smooth manifolds, in fact there is a general criterion whether we have this nice correspondence for any locally ringed space. First we have to do a bit of bookkeeping but it's worth it I swear. First some notation, this is taken from [2]

Definition 5.1.

1. We call the category of $\mathcal{O}_X$-modules $\mathcal{O}_X - \mathbf{mod}$
2. We call the full subcategory of locally free $\mathcal{O}_X$ modules of finite rank $\mathbf{Lfb}(X)$
3. We call the full subcategory of $A - \mathbf{mod}$ consisting of finitely generated A modules $\mathbf{Fgp}(A)$

Definition 5.2 (Acyclic). We say that a sheaf $\mathcal{F}$ is *acyclic* if all higher sheaf cohomology groups vanish, these are the groups constructed by the right derived functor of the global sections functor, by higher we mean $H^i(X, \mathcal{F}) = 0$ for $i \geq 1$

Definition 5.3 (Generated by Global Sections). We say that an $\mathcal{O}_X$ module is *generated by global sections* if there is a family of sections $\{s^i\}_{i\in I}$ in $\mathcal{F}(X)$ such that for each $x \in X$, the set of $\{s^i_x\}_{i\in I}$ generate $\mathcal{F}_x$ as an $\mathcal{O}_x - module$. We say that a sheaf is finitley generated by global sections if I is finite

Definition 5.4. Let $(X, \mathcal{O}_X)$ be a locally ringed space. We say a subcategory $\mathcal{C}$ of $\mathcal{O}_X - \mathbf{mod}$ is called an *admissible subcategory* if it satisfies the following

1. $\mathcal{C}$ is a full abelian subcategory of $\mathcal{O}_X - \mathbf{mod}$ and $\hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are in $\mathcal{C}$ whenever $\mathcal{F}, \mathcal{G}$ are in $\mathcal{C}$ ($\hom_{\mathcal{O}_X}$ refers to the sheaf of $\mathcal{O}_X$ morphisms)
2. Every sheaf in $\mathcal{C}$ is *acyclic* and *generated by global sections*
3. $\mathbf{Lfb}(X)$ is a full subcategory of $\mathcal{C}$

We can now state the most general Serre-Swan criterion I am aware of

Theorem 5.2. *Let* $(X, \mathcal{O}_X)$ *be a locally ringed space, and let* $A = \mathcal{O}_X(X)$*. if* $\mathcal{O}_X - \mathbf{mod}$ *contains an admissible subcategory* $\mathcal{C}$*, and every sheaf in* $\mathbf{Lfb}(X)$ *is finitely generated by global sections. Then the global sections functor is a categorical equivalence* $\mathbf{Lfb}(X) \to \mathbf{Fgp}(A)$*. We have the Serre-Swan correspondence for* $(X, \mathcal{O}_X)$

6 Some Geometry

Now we are just gonna do some cool algebraic geometry found in the exercises of Hartshorne [1]. I suppose this is becoming some sort of master doc for alg-geom.

One of the main things you want to look at in geometry and number theory is finding integer or rational points on curves. This exemplified by

Theorem 6.1 (Fermat's Last Theorem)**.** *The curve $x^n + y^n = 1$ has no rational points for $n > 2$ other than those with $xy = 0$*

If we have some bog-standard algebraic variety, say $R = \frac{\mathbb{Z}[x,y]}{(x^2+y^2-1)}$ then a rational point on this variety is a choice of rational x, y such that $x^2+y^2-1 = 0$. I.e. it is a ring morphism $R \to \mathbb{Q}$. Or equivalently it is a morphism of affine schemes $\operatorname{spec} \mathbb{Q} \to \operatorname{spec} R$. Where since $\mathbb{Q}$ is a single point this morphism is just picking out a certain point. We can do this for general schemes, if we define a $K-$point of a scheme X to be a morphism $\operatorname{spec} K \to X$ then the data of a $K-$point is just some point $x \in X$ and an inclusion map from the residue field $k(x) = \mathcal{O}_x/\mathfrak{m}_x$, $k(x) \to K$.

We can prove this quite easily, since the morphism must send the unique point of $\operatorname{spec} K$ somewhere that gives us a point, we then just need to define the corresponding map of sheaves. For some $U \subset X$, if $x \notin U$ then the map is just to the zero ring so we have nothing to do, if $x \in U$ then we have some map to K that agrees with restrictions. Since we have a family of such maps this gives a natural transformation from $\mathcal{O}_x$ restricted to sets over x to the constant functor ΔK, this is equivalent to giving a morphism $f : \mathcal{O}_X \to K$ which since this is just the induced map of stalks it is a local homomorphism so $\mathfrak{m}_x$ must map to the maximal ideal of K which is zero so this is equivalent to giving a map $\mathcal{O}_X/\mathfrak{m}_x \to K$.

When calculus was first invented mathematicians would use the concept of an infinitesimal, a number so small that once you square it it's gone. In algebraic geometry then there is a nice link between the concept of derivatives and this new type of number.

We define the dual numbers over a field k as $D = k[\varepsilon]/\varepsilon^2$, our elements are elements of k with some infinitesimal shift added. This is an algebra over k so we have a natural inclusion $k \to D$. This means we have a natural map $\operatorname{spec} k[\varepsilon]/\varepsilon^2 \to \operatorname{spec} k$ so we can consider this as a scheme over k. If then we have another scheme over k, call it X. Then we say a k-morphism is a map between these which agrees with the map onto $\operatorname{spec} k$ since $k[\varepsilon]/\varepsilon^2$ has a unique prime ideal given by (ε) any map out of it picks out a point, say $x \in X$, passing then to stalks we see that we have a commutative triangle

$$
\begin{array}{ccc}
\operatorname{spec} k[\varepsilon]/\varepsilon^2 & \longrightarrow & X \\
\downarrow & \swarrow & \\
\operatorname{spec} K & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
k[\varepsilon]/\varepsilon^2 & \longleftarrow & \mathcal{O}_x \\
\uparrow & \nearrow & \\
k & &
\end{array}
$$

So $\mathcal{O}_x/\mathfrak{m}_x \cong k$ since it must be a subfield of k and must contain k. Additionally since $f_p^{\#}$ is a local homomorphism we see that we can induce a map $\mathfrak{m}_x/\mathfrak{m}_x^2 \to (\varepsilon)/(\varepsilon)^2 \cong k$ so an element of the dual of the cotangent space $\mathfrak{m}_x/\mathfrak{m}_x^2$ aka the tangent space. Additionally if x is rational we can define the map sending the unique ideal to x and given an element of the tangent space ϕ we just want to construct our map of sheaves, given $x \notin U$ we are mapping to the zero ring so we are done already, and if $x \in U$ we see from before that it's sufficient to find a map $\mathcal{O}_x \to k[\varepsilon]/\varepsilon^2$ by sending a point since $\mathcal{O}_x/\mathfrak{m}_x \cong k$ every element can be written as $a + b$ for $a \in k$, $b \in \mathfrak{m}_x$, we then send $a + b \to a + \phi(\bar{b})\varepsilon$ to get our sheaf morphism. This is all to say that elements of the tangent space at $k-$rational points are just $k-$morphisms from the dual numbers

References

[1] Robin Hartshorne. *Algebraic geometry*. 1 1977.

[2] Archana S. Morye. Note on the Serre-Swan theorem. *arXiv (Cornell University)*, 1 2009.