Theory of Group Stacks

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Contents

0.1 Abstract

This document is intended as an introduction to stacks and group stacks. Building on ideas from algebraic geometry and category theory we treat foundational questions in the subject focusing on the categorical aspects in Chapter 1 and Chapter 3 and on the geometric in Chapter 2. Using this foundation then we give fundamental examples of group stacks in Chapter 4 and Chapter 5 giving a treatment of when we can expect a quotient to give us a group stack and how the application of the Picard stack can be used to attack a fundamental moduli problem in geometry.

0.2 Introduction

In geometry we study spaces, one question that arises is what the correct type of space is to study. One such answer is stacks, a way to treat a space as a special type of functor. These allow us to solve more general problems than other types of space such as finding the moduli space of eliptic curves. Another thing they allow us to do is to take quotients, the motivation begind the study of group stacks then is that commonly in geometry we have group schemes acting on group schemes and it would be useful to be able to take the quotient. We discuss when this quotient inherits the group structure in a natural way so that the theory of 2-groups can be applied when stack-less approaches fail.

In Chapter 1 we cover the foundational aspects of stacks, building up some theory of higher categories in order to define stacks as a 2-categorical analogue of a sheaf.

In Chapter 2 we treat the main topologies used in the theory of stacks. Explaining the reasons they're used along with the types of stacks that can be treated geometrically using these topologies. In addition to this there is some mention of uses of stacks outside of algebraic geometry, namely in differential geometry with a description of Lie Groupoids

Chapter 3 then focuses back on the categorical aspects of group stacks, internalising the idea of a group into a 2-category to create 2-groups and so creating the object of focus, group stacks. There is also some discussion on how one would generalise this construction to higher categories

Chapter 4 covers the main concrete example of a stack, the quotient stack. Allowing us to describe when this quotient can be treated geometrically in spite of it not existing as a scheme. This is then used to give a family of examples of group stacks as well as due to Deligne a full classificaition of abelian group stacks.

To conclude, in Chapter 5 we covers the classical constructions around the Picard group. To do so introducing Čech cohomology, as well as discussion of a result of Grothendieck describing the existance of the Picard scheme. This is then extended to give a very powerful example of a group stack the Picard stack and how this group stack can be used to recover the Picard scheme.

Chapter 1

What is a Stack?

1.1 Sheaves

The answer to the question "What is a Stack?" is actually very easy, a stack is a 2-sheaf of groupoids on $\mathbf{Sch}/S$ with some topology. This of course is meaningless unless we know how to generalise sheaves to this degree. So we start there.

Speaking classically, a sheaf on a space X is a functor $F : \mathrm{Ouv}(X) \to \mathbf{Set}^{op}$ such that for any open cover $\{U_\lambda \to U\}_{\lambda\in\Lambda}$ we have an equaliser diagram

$$F(U) \to \prod_{\lambda\in\Lambda} F(U_\lambda) \rightrightarrows \prod_{\mu,\lambda\in\Lambda} F(U_\lambda \cap U_\mu)$$

Where the two arrows are $(f_\lambda) \mapsto (f_\lambda|_{U_\lambda\cap U_\mu})$ and $(f_\lambda) \mapsto (f_\mu|_{U_\lambda\cap U_\mu})$. Speaking categorically $U_\lambda \cap U_\mu$ gives us the infimum of U, V which is the categorical product, but since we have this whole setup relative to U this is equivalently the categorical fibered product $U_\lambda \times_U U_\mu$

$$\begin{tikzcd}
& U & \\
U_\lambda \arrow[ur] & & U_\mu \arrow[ul] \\
& U_\lambda \cap U_\mu = U_\lambda \times_U U_\mu \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

so the sheaf condition is maybe better written as the following equaliser

$$F(U) \to \prod_{\lambda\in\Lambda} F(U_\lambda) \rightrightarrows \prod_{\mu,\lambda\in\Lambda} F(U_\lambda \times_U U_\mu)$$

Where the two maps are induced by the projection maps from the fibered product. Now that we have generalised enough it is somewhat obvious how we generalise this to an arbitrary category, we just want coverings and fiber products.

Definition 1.1.1 (*Grothendieck Topology*). [LABG] A Grothendieck topology [1] on a category C is a collection of families of maps $\{\phi_\lambda \colon U_\lambda \to U\}_{\lambda\in\Lambda}$ called coverings, these are the analogies of open covers. This collection, call it T satisfies the following

1. For any isomorphism ϕ, $\{\phi\} \in T$
2. If $\{U_\lambda \to U\}_{\lambda\in\Lambda} \in T$ and for each λ, $\{V_{\lambda,\mu} \to U_\lambda\}_{\mu\in M} \in T$ then the compositions $\{V_{\lambda,\mu} \to U_\lambda \to U\}_{(\lambda,\mu)\in\Lambda\times M} \in T$
3. If $\{U_\lambda \to U\}_{\lambda\in\Lambda} \in T$ and $V \to U$ is any morphism then $U_\lambda \times_U V$ exist and $\{U_\lambda \times_U V \to V\}_{\lambda\in\Lambda} \in T$

A category with a Grothendieck Topology is called a site. Some examples are

1. For a topological space X we have the site of open sets $\mathrm{Ouv}(X)$, which is the category of open sets along with coverings $\{U_\lambda \to U\}_{\lambda\in\Lambda}$ when $U = \bigcup_\lambda U_\lambda$
2. We can go one step up and consider the category of topological spaces **Top** as a site taking coverings $\{\iota_\lambda \colon U_\lambda \to X\}_{\lambda\in\Lambda}$ where each ι_λ is an open immersion and $X = \bigcup_\lambda \iota_\lambda U_\lambda$
3. In the context of algebraic geometry we can choose $\tau-$coverings for $\tau \in \{$fppf,smooth, étale, Zariski$\}$. That is coverings as before where we restrict ι to satisfy property τ. Then for any scheme S we can define the big $\tau-$site of S, $(\mathbf{Sch}/S)_\tau$ as the slice category $\mathbf{Sch}/S$ and $\tau-$coverings, these objects will be the focus of Chapter 2

In the exact same way as before we can consider the sheaves on this site.

Definition 1.1.2 (*Sheaf on a site*). We define a sheaf on a general site X, T to be a functor $X \to \mathbf{Set}^{op}$ so that for any covering family $\{U_\lambda \to U\}_{\lambda\in\Lambda} \in T$ the following diagram is an equaliser

$$F(U) \to \prod_{\lambda\in\Lambda} F(U_\lambda) \rightrightarrows \prod_{\mu,\lambda\in\Lambda} F(U_\lambda \times_U U_\mu)$$

A question one may have at this point is why are we doing all of this. In algebraic geometry we care about moduli problems, that is we care about representing functors with schemes X so that the S-points of X parameterise $F(S)$, ie $\hom(-, X) \cong F(-)$.

Example 1.1.3. Suppose we have a locally small category with any many limits/colimits as we want, Consider the presheaf $\hom(-, X)$. Suppose for some U we have a collection $\bigsqcup_\lambda U_\lambda \to U$ that informally covers U, ie U is just this collection of U_λ where each pair is glued together along some map, we can represent this as taking the pullbacks of the maps $U_\lambda \to U$. That is we have a coequaliser

[1] this is sometimes called a Grothendieck Pretopology but this is good enough for now to give it the full title

$$U \longleftarrow \coprod_{\lambda\in\Lambda} U_\lambda \leftleftarrows \coprod_{\mu,\lambda\in\Lambda} U_\mu \times_U U_\lambda$$

Then by continuity of $\hom(-,X)$ we have the limit diagam

$$\hom(U,X) \longrightarrow \hom\left(\coprod_{\lambda\in\Lambda} U_\lambda, X\right) \rightrightarrows \hom\left(\coprod_{\mu,\lambda\in\Lambda} U_\mu \times_U U_\lambda, X\right)$$

Where thanks to universal properties we have an isomorphism of diagrams

$$\begin{tikzcd}
\prod_{\lambda\in\Lambda} \hom\left(U_\lambda, X\right) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & \prod_{\mu,\lambda\in\Lambda} \hom\left(U_\lambda \times U_\mu, X\right)) \arrow[d, "\cong"] \\
\hom\left(\coprod_{\lambda\in\Lambda} U_\lambda, X\right) \arrow[r, shift left] \arrow[r, shift right] & \hom\left(\coprod_{\mu,\lambda\in\Lambda}, U_\lambda \times_U U_\mu, X\right)
\end{tikzcd}$$

So $\hom(U,X)$ exists as the limit of

$$\prod_{\lambda\in\Lambda} \hom(U_\lambda, X) \rightrightarrows \prod_{\mu,\lambda\in\Lambda} \hom(U_\lambda \times_U U_\mu, X)$$

That is to say, for any reasonable site[2] we would expect representable functors to be sheaves

So we see that, in most natural cases, being a representable functor is a special case of being a sheaf. Thanks to the Yoneda lemma one can replace objects with representable functors without losing any information. This allows us to treat sheaves as a more general type of object in a site.

If we allow for just any old sheaf however, our moduli problem becomes sort of boring. Since we can just check when the functor itself is a sheaf, we have generalised beyond application. Our original aim was to study these families geometrically. To do so we look at those sheaves that are, in some sense, locally just schemes. The definition of this can wait until we have covered the actual topologies one can give to the category of schemes (see Chapter 2). Nonetheless we call such sheaves Algebraic Spaces.

The theory of Algebraic Spaces is expansive and can be used to solve many problems in algebraic geometry. Sadly they come with a fundamental flaw, they forget too much about the objects. Say we are looking at the moduli space for some family of curves. Algebraic spaces can get as far as saying that for an equivalence class of curves over $\mathbb{Q}[x]$ we get a $\mathbb{Q}[x]$ point of our space. This is fine for some cases but just because two curves are isomorphic doesn't mean we can ignore their differences. That is to say there can be multiple ways in which the curves are isomorphic and combining them all into one class requires you to forget that this is the case and just try to take some canonical isomorphism down to one representative. An example of where this issue arises can be found in the introduction of [OLSS] wherin a family of curves that are not isomorphic are made isomorphic by a field

[2]The technical term here is subcanonical, topologies finer than the canonical topology. The canonical topology is then the largest topology where $\hom(-,X)$ is always a sheaf[ARTI]

extension. Since looking for just a moduli scheme we must forget in what way they're isomorphic these points get glued together, contradicting that for a scheme taking a field extension only adds points.

This is a problem when trying to find moduli spaces, so to get around it we just want to replace our measly sets with something that remembers the isomorphisms, that is to say we use groupoids.

1.2 Stacks

Since groupoids are in and of themselves categories, the category of all groupoids looks like a category of categories. So to treat Stacks properly we need to consider natural isomorphisms and generally the higher dimensional structure that groupoids come with. To do so we first flesh out the notion of a 2-category.

Definition 1.2.1 (*2-category*)**.** [3] A 2-category $\mathcal{C}$ is a category enriched over **Cat**, ie for each $x, y \in \mathcal{C}$ we have a category $\hom(x, y)$, then for any triple x, y, z we have a composition functor $\circ : \hom(x, y) \times \hom(y, z) \to \hom(x, z)$ We say that x, y are objects. Then objects of $\hom(x, y)$ are morphisms for morphisms f, g an element of $\hom(f, y)$ are 2-morphisms. If every 2-morphism is invertable then we say that $\mathcal{C}$ is a $(2, 1)$-category. For example our main case of **Gpd** is a $(2, 1)$-category

Note that any category is naturally a 2-category by replacing each hom set with the corresponding discrete category.

Now, as always in mathethematics once we have an object we want to consider how to map between these objets, there are a few approaches but the one we use for stacks is that of pseudofunctors

Definition 1.2.2 (*Functors*)**.** For 2-categories $\mathcal{C}, \mathcal{D}$ a pseudofunctor $F : \mathcal{C} \to \mathcal{D}$ or for simplicity we may call it a functor. Is comprised of the following data

1. For each object $x \in \mathcal{C}$ an object $F(x) \in \mathcal{D}$
2. For each hom category in $\mathcal{C}$ a functor $\overrightarrow{F} : \hom(x, y) \to \hom(F(x), F(y))$
3. For each object $x \in \mathcal{C}$ an isomorphism $F_{\mathrm{id}} : \mathrm{id}_x \to \overrightarrow{F}(\mathrm{id}_x)$
4. For each triple of objects $x, y, z \in \mathcal{C}$ and pair of morphisms $f \in \hom(x, y)$, $g \in \hom(y, z)$, an isomorphism $\overrightarrow{F}(f) \circ \overrightarrow{F}(g) \to \overrightarrow{F}(f \circ g)$ natural in f, g

Such that these isomorphisms are coherent, that is any way to turn n-ary compositions in $\mathcal{C}$ into n-ary compositions in $\mathcal{D}$ results in the same isomorphism

[3]These are often called strict 2-categories but the only one we care about for now is groupoids so we don't need to think too hard about it

If $\mathcal{C}$ or $\mathcal{D}$ are dual categories of the categories we really care about we call such an object a pre-stack

In order to construct sheaves with these sorts of categories we must first define what limits are in these categories. While there are more complicated types we only need the most basic kind, recall for standard categories that a limit is an object so that any map into a diagram factors uniquely through it, for 2-categories we do the same thing except instead of uniquely factoring, we factor uniquely up to unique 2-morphism, this is again just saying that the object holds all of the information about the diagram but in the natural way for 2-categories[4].

Sheaves were first defined to generalise taking the set of functions. So in order to do this for higher categories we want an analogue of functions that requires this categorical structure. To do so we consider the following

Example 1.2.3. For a topological space X we can consider the functor

$$\mathrm{Vect}_{\mathbb{R}} : \mathrm{Ouv}(X)^{op} \to \mathbf{Gpd}$$

sending an open set to the groupoid of vector bundles over this set.

We take this as the motivating example for 2-sheaves, or as we will call them, stacks. When we were considering 1-sheaves we note that to glue functions you need on each intersection, equalities $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Now the reason vector bundles require some higher dimensional structure is that in order to glue we want not equalities but isomorphisms $\phi_{ij} : V_i|_{U_i \cap U_j} \tilde{\to} V_j|_{U_i \cap U_j}$ where on each triple intersection $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$

That is to say a vector bundle is determined by the restriction to each intersection and then on each intersection a collection of coherent isomorphisms between the restrictions corresponding to each inclusion $U_i \cap U_j \cap U_k \to U_i \cap U_j$ so in some sense our functor equalises the diagram

$$F(U) \to \prod_{\lambda \in \Lambda} F(U_\lambda) \rightrightarrows \prod_{\lambda, \mu \in \Lambda} F(U_\lambda \times_U U_\mu) \overset{\to}{\rightrightarrows} \prod_{\lambda, \mu, \nu \in \Lambda} F(U_\lambda \times_U U_\mu \times_U U_\nu)$$

And it is from this we define our higher dimensional sheaves.

In standard category theory we can give explicit descriptions of many limits by writing them as some composition of products and equalisers allowing for more hands on approaches to the objects in question. As luck would have it this is true in many 2-categories. In our case the base 2-category that we are dealing with is that of groupoids so it is worthwhile to give an explicit description of a limit in this case.

[4]One perspective on why this is natural is that in higher category theory the "correct" notion of uniqueness is contractability. If we contract the ∞-categorical notion of contractibility we get just unique natural transformations.

Lemma 1.2.4 (*Limit in groupoids*)**.** *[GROE] For a diagram $C_{-} : I \to \mathbf{Gpd}$. We define the limit $\lim_{i \in I} C_i$ as the groupoid whos objects are collections $\{X_i \in C_i\}_{i \in I}$ and for each morphism $\alpha : i \to j$ in I, we have an isomorphism $\phi_\alpha : C_\alpha(X_i) \tilde{\to} X_j$ that agrees with composition, ie $\phi_{\alpha \circ \beta} = \phi_\alpha \circ \phi_\beta$. We can see that this satisfies the universal properties we would expect in the category of groupoids.*

This finally gives us enough information to define stacks

Definition 1.2.5 (*Stack*)**.** A prestack $F : \mathcal{C}^{op} \to \mathbf{Gpd}$ for a site $(\mathcal{C}, \mathcal{T})$ is a stack if for each covering $\{U_\lambda \to U\}_{\lambda \in \Lambda}$ if the diagram

$$F(U) \to \prod_{\lambda \in \Lambda} F(U_\lambda) \rightrightarrows \prod_{\lambda, \mu \in \Lambda} F(U_\lambda \times_U U_\mu) \rightrightarrows\!\!\!\!\to \prod_{\lambda, \mu, \nu \in \Lambda} F(U_\lambda \times_U U_\mu \times_U U_\nu)$$

induces a limit in $\mathbf{Gpd}$. Where the maps are induced by the diagram

$$\begin{tikzcd}
& U & \\
U_\lambda \arrow[ur] & U_\mu \arrow[u] & U_\nu \arrow[ul] \\
U_\lambda \times_U U_\mu \arrow[u] \arrow[ur] & U_\lambda \times_U U_\nu \arrow[ul] \arrow[ur] & U_\mu \times_U U_\nu \arrow[ul] \arrow[u] \\
& U_\lambda \times_U U_\mu \times_U U_\nu \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Now while this definition is clean it is somewhat hard to work with since it requires thinking 2-categorically meaning you need to consider morphisms and objects at the same time and worry about their interactions. In this specific case however we can separate these two modes of thought thanks to the following lemma

Lemma 1.2.6. *[Slogan: You can glue objects and glue morphisms] [GROE]A prestack*

$$F : (\mathcal{C}, \mathcal{T})^{op} \to \mathbf{Gpd}$$

is called a stack if and only if the following 2 conditions hold

1. *For each $\{U_\lambda \to U\}_{\lambda \in \Lambda}$ and a collection of object $\{X_\lambda \in F(U_\lambda)\}_{\lambda \in \Lambda}$ with isomorphisms*

   $$\phi_{\lambda\mu} : X_i|_{U_\lambda \times_U U_\mu} \tilde{\to} X_j|_{U_\lambda \times_U U_\mu}$$

   that satisfy the cocycle condition $\phi_{\lambda\mu} \circ \phi_{\mu\nu} = \phi_{\lambda\nu}$ on $U_\lambda \times_U U_\mu \times_U U_\nu$. There is some $X \in F(U)$ with isomorphisms on each U_λ, $\phi_\lambda : X|_{U_\lambda} \tilde{\to} X_\lambda$. Ie we can glue together the objects X_i if they agree up to isomorphism on intersections.

2. *For each $U \in \mathcal{C}$ $X, Y \in F(U)$, the presheaf $h(X, Y) : C/U \to \mathbf{Set}$, defined by $h(X, Y)(V \to U) = \hom_{F(V)}(X|_V, Y|_V)$ is a sheaf.*

Proof. ($\implies$) For the first direction, Suppose a prestack F satisfies (1.) and (2.). Fixing a covering $\{U_\lambda \to U\}_{\lambda\in\Lambda}$ we see that since we have the diagram

$$F(U) \to \prod_{\lambda\in\Lambda} F(U_\lambda) \rightrightarrows \prod_{\lambda,\mu\in\Lambda} F(U_\lambda \times_U U_\mu) \substack{\to\\\to\\\to} \prod_{\lambda,\mu,\nu\in\Lambda} F(U_\lambda \times_U U_\mu \times_U U_\nu)$$

There is a natural map from $F(U)$ to the actual limit, L, of

$$L \to \prod_{\lambda\in\Lambda} F(U_\lambda) \rightrightarrows \prod_{\lambda,\mu\in\Lambda} F(U_\lambda \times_U U_\mu) \substack{\to\\\to\\\to} \prod_{\lambda,\mu,\nu\in\Lambda} F(U_\lambda \times_U U_\mu \times_U U_\nu)$$

We want to show that this map is an equivalence. Since we are in **Gpd** $\subset$ **Cat** it is sufficient to show that this functor is fully faithful and essentially surjective.

First, we define our notation, we name the maps

$$a_1, a_2 : \prod_{\lambda\in\Lambda} F(U_\lambda) \rightrightarrows \prod_{\mu,\lambda\in\Lambda} F(U_\lambda \times_U U_\mu)$$

$$b_1, b_2, b_3 : \prod_{\mu,\lambda\in\Lambda} F(U_\lambda \times_U U_\mu) \substack{\to\\\to\\\to} \prod_{\nu,\mu,\lambda\in\Lambda} F(U_\lambda \times_U U_\mu \times_U U_\nu)$$

First we prove essential surjection. The limit L has an explicit description of its objects by Theorem 1.2.4 so take an arbitrary such object (X, Y, Z)

$$X \in \prod_{\lambda\in\Lambda} F(U_\lambda), Y \in \prod_{\mu,\lambda\in\Lambda} F(U_\lambda \times_U U_\mu), Z \in \prod_{\nu,\mu,\lambda\in\Lambda} F(U_\lambda \times_U U_\mu \times_U U_\nu)$$

With isomorphisms $\zeta_n : Y \tilde{\to} a_n(X)$, $\xi_m : Z \tilde{\to} b_m(C_2)$ that compose to give isomorphisms $Z \tilde{\to} b_m a_n(X)$. Our isomorphisms ζ_n give isomorphisms $\zeta_n^{\lambda\mu} : Y_{\lambda\mu} \to X_\lambda|_{U_\lambda \times_U U_\mu} = a_n(X)_{\lambda\mu}$ by the definition of products of groupoids. This means we get isomorphisms

$$\phi_{\lambda\mu} = \zeta_2^{\lambda\mu}(\zeta_1^{\lambda\mu})^{-1} : X_\lambda|_{U_\lambda \times_U U_\mu} \tilde{\to} X_\mu|_{U_\lambda \times_U U_\mu}$$

These trivially satisfy the coycycle condition. Since each $X_\lambda \in F(U_\lambda)$ by the assumptions there is some $\tilde{X} \in F(U)$ with isomorphisms $\tilde{X}|_{U_\lambda} \to X_\lambda$ so $\tilde{X} \mapsto X' \cong (X, Y, Z)$ so the map is essentially surjective.

To show that the map too is fully faithful we take some morphism

$$(f_1, f_2, f_3) : (C_1, C_2, C_3) \to (C_1', C_2', C_3')$$

in L. These morphisms give commutative diagrams

$$
\begin{tikzcd}
C_2 \arrow[r, "\cong"] \arrow[d, "f_2"] & a_n(C_1) \arrow[d, "a_n(f_2)"] \\
C_2' \arrow[r, "\cong"] & a_n(C_1')
\end{tikzcd}
\qquad
\begin{tikzcd}
C_3 \arrow[r, "\cong"] \arrow[d, "f_3"] & b_m(C_2) \arrow[d, "a_n(f_3)"] \\
C_3' \arrow[r, "\cong"] & b_m(C_2)
\end{tikzcd}
$$

So morphisms are determined just by what happens at C_1 and must satisfy $\alpha_1(f_1) = \alpha_2(f_2)$ by the property of L being a limit. Thus by letting $C_1 = X$, we see that in general morphisms in L are just morphisms in the equaliser of

$$\hom(X|_{U_\lambda}, X'|_{U_\lambda}) \rightrightarrows \prod_{\lambda\mu\in I} \hom(X|_{U_\lambda\times_U U_\mu}, X'|_{U_\lambda\times_U U_\mu})$$

Since $h(X,Y)$ is a sheaf we have that this is exactly $\hom(X,X')$ so our functor is fully faithful. This means that $F(U)$ is equivalent to the limit L so is a stack.

($\Longleftarrow$) Suppose that F is a stack. Then for every covering $\{U_\lambda \to U\}_{\lambda\in\Lambda}$, $F(U)$ is isomorphic to the limit of the diagram

$$\prod_{\lambda\in\Lambda} F(U_\lambda) \rightrightarrows \prod_{\lambda,\mu\in\Lambda} F(U_\lambda \times_U U_\mu) \rightrightarrows\!\!\!\!\!\!\rightarrow \prod_{\lambda,\mu,\nu\in\Lambda} F(U_\lambda \times_U U_\mu \times_U U_\nu)$$

So for a collection $\{X_\lambda \in F(U_\lambda)\}_{\lambda\in\Lambda}$ and isomorphisms $\phi_{\lambda\mu} : X_\lambda|_{U_\lambda\times_U U_\mu} \to X_\mu|_{U_\lambda\times_U U_\mu}$ satisfying the cocycle condition we get an object in the limit

$$\begin{aligned} F(U) \ni X &\cong \big((X_\lambda)_{\lambda\in\Lambda}, (X_\lambda|_{U_\lambda\times_U U_\mu})_{\mu,\lambda\in\Lambda}, (X_\lambda|_{U_\lambda\times_U U_\mu\times_U U_\nu})_{\nu,\mu,\lambda\in\Lambda},\big) \\ &=: (X^1, X^2, X^3) \end{aligned}$$

By construction then there are isomorphisms $\phi_i : X|_{U_\lambda} \to X_\lambda$ so (1.) is satisfied. To then satisfy (2.) we take some collection $\{U_\lambda \to U\}_{\lambda\in\Lambda}$ in C/U. Then for each $(f_\lambda)_{\lambda\in\Lambda}$ in the equaliser of

$$\prod_{\lambda\in\Lambda} \hom(X|_{U_\lambda}, Y|_{U_\lambda}) \rightrightarrows \prod_{\mu,\lambda\in\Lambda} \hom(X|_{U_\lambda\times_U U_\mu}, Y|_{U_\lambda\times_U U_\mu})$$

We obtain a morphism of the $(X^1, X^2, X^3) \to (Y^1, Y^2, Y^3)$ as before and so we get a unique map $X \to Y$ that restricts to each $X|_{U_\lambda} \to X|_{U_\lambda}$. This is exactly saying that

$$\hom(X,Y) \to \prod_{\lambda\in\Lambda} \hom(X|_{U_\lambda}, Y|_{U_\lambda}) \rightrightarrows \prod_{\mu,\lambda\in\Lambda} \hom(X|_{U_\lambda\times_U U_\mu}, Y|_{U_\lambda\times_U U_\mu})$$

Is an equaliser. That is to say that $h(X,Y)$ is a sheaf so (2.) must too be satisfied. $\square$

We have now constructed stacks as a generalised type of sheaf. Recall from earlier that we constructed sheaves as a functor and had a problem, we have lost the geometry. I teased that we would define so called "Algebraic Spaces" to bring the geometry back but for that we needed to consider the topology on $(\mathbf{Sch}/S)$. We can do a similar thing for stacks, looking for stacks that are locally geometric with respect to different topologies. For that we need to find some nice topologies, so lets do that.

Chapter 2

Topologies

2.1 The Zariski Topology

We start in the simplest category we can, that of affine schemes $\mathbf{Aff} \cong \mathbf{CRing}^{op}$. For rings, say $X = \operatorname{spec} R$ a basic open subset $X_f \subset X$ corresponds to those primes not containing f. THis is precicely the primes of R_f so the induced map $\operatorname{spec} R_f \to \operatorname{spec} R$ is an open mapping. From this we can define a topology

Definition 2.1.1 (*Affine Zariski Site*)**.** For the category of affine schemes, or equivalently the opposite category of commutative rings we define a site by saying that a family $\{\phi_\lambda : \operatorname{spec} A_\lambda \to \operatorname{spec} R\}_{\lambda\in\Lambda}$ is a covering if

1. Each A_i is the localisation $R_{f_\lambda} \cong f_\lambda^{-1}R$ for some $f \in R$. Ie $\operatorname{spec} A_\lambda$ is a basic open set of $\operatorname{spec} R$
2. The map ϕ is the dual of the canonical map $R \to R_f$. Ie ϕ is the inclusion map of $X_f \to X$
3. For some $\lambda_1 ... \lambda_n$ there exists $a_1 ... a_n$ so that $\sum_i a_i f_{\lambda_i} = 1$. Ie the open sets X_f cover $\operatorname{spec} R$

We can see that this site embeds simply into the category of schemes Sch and so we should expect that for any scheme, the functor of points $\hom_{Sch}(-, S) : \mathbf{Aff} \to \mathbf{Set}$ is a sheaf on this site. Additionally this is a very geometric type of sheaf, in fact this determines the scheme itself, so it is important to look at what makes this functor so important.

The important thing about schemes is that they locally look like affine schemes. That is there is a map $\bigsqcup \operatorname{spec} A_i \to S$ that is surjective and locally an isomorphism. There is then the additional property that the affine covers of a scheme determine its properties. That is if we have some morphism of schemes $S \to X$ and we want to say that it has some property, we could check this by looking at wether the property holds on the affine covers. In order to do so however we need the data of the morphism to be remembered by the affine subsets. This is called being representable by affine schemes. Put more rigourously $S \to X$ has property P whenever there is an affine scheme A with a morphism $A \to X$ so that $S \times_X A$ is affine and the morphism has property P. Ie $S \times_X A$ is the affine subscheme on which property P holds.

Definition 2.1.2. Let $\mathcal{C}$ be a site, then for a morphism $f : F \to G$ of presheaves we say that f is representable by objects of type T if for all hom sheaves $\text{よ}X$ and morphims $\text{よ}X \to G$, the fiber product $F \times_G \text{よ}X$ is of type T

To then be able to treat stacks and sheaves locally we want to have the ability to factor things locally. So for a genreral geometric object we want all maps $\text{よ}X \to F$ to be representable. There is a nice way to summerise this

Proposition 2.1.3. *If the diagonal map $F \to F \times F$ is representable, then any map $S \to F$ is representable.*

And so to ask wether the object can be treated locally it suffices to ask wether its diagonal morphism is nice. This formulation being more intrinsic to the object itself

2.2 The fppf Topology

One strange and somewhat technical things used in the study of algebraic stacks is the fppf topology. Being a loan acronym from french, fppf stands for "fidèlement plat de présentation finie" meaning "faithfully flat of finite presentation". And so in definition it is very straightforward

Definition 2.2.1. A family of maps $\{\phi_\lambda : U_\lambda \to X\}_{\lambda\in\Lambda}$ is a covering in the fppf topology if each morphism ϕ_λ is flat and locally finite presentation and $X = \bigcup \phi_\lambda(U_\lambda)$

Note that the fppf property is well behaved under base change and composition so this is infact a topology. The reason for using this topology is somewhat less intuitive than the others, since étale and smooth morphisms have very nice geometric meanings whereas this fppf topology lacks some of the geometric richness. The advantage of this however is that it deals nicer with the idea of treating properties locally[1]. For example if we are looking at to morphisms $X \to S, Y \to S$ with a morphism $f : X \to Y$ making this commute, if f can be split up into a cover in the fppf topology then $X \to S$ being flat, locally finite type or locally finite presentation means that $Y \to S$ is too so it allows us to move the local properties of X through to local properties of Y. In addition to this in a more topological manner it allows us to cut up our space into chunks that let us check wether a map is an immersion. That is if $\{\phi_\lambda : U_\lambda \to Y\}_{\lambda\in\Lambda}$ is an fppf cover then a map $f : X \to Y$ is an immersion if and only if the maps $X \times_Y U_i \to U_i$ are. So the "allowed open sets" for the fppf topology are in some sense the right size for defining properties locally. Since

[1] The following facts come from Stacks project 35.14.8 and 35.24.1

we want to deal with stacks as we would with schemes, by defining properties locally and then extending them, this fact of the fppf topology is clearly very useful.

2.3 The Étale Topology

Étale morphisms are one of the most important objects in algebraic geometry. They are the algebraic analogue of local isomorphisms. In analytic geometry the standard result for local isomorphisms is the implicit function theorem and so we require a similar setup for étale morphisms

Theorem 2.3.1. *For a smooth function $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$ then for a point x with $f(x) \neq 0$ if the Jacobian is non singular then the set $\{f = 0\}$ is locally a graph, that is it is parameterised as some $(p, g(p))$ for $p \in \mathbb{R}^n, g : \mathbb{R}^n \to \mathbb{R}^m$*

This motivates the canonical example of an étale morphism, that of a graph. Suppose we have some graph $y = f(x)$ then we should want the projection map $(x, y) \mapsto x$ to be a local homeomorphism. In fact any function $f(x, y) = 0$ should be a local homeomorphism as long as it is not vertical, that is the derivative $\partial_y f(x, y)$ is not zero. This construction gives us so called “standard étale” morphisms once we make it more about rings

To make this about rings we replace the set $f(x, y) = 0$ with the ring $A[y]/f$ and then when we restrict to the places where $f' \neq 0$ we are looking at some open subset, for this we can just take a principal open and so we are looking at this localised at some g so our standard example should be that of the ring $(A[y]/f)_g$

Definition 2.3.2. If B is an $A-$algebra, then say we can choose $f, g \in A[y]$ so that f' is invertable in $(A[y]/(f))_g$. We say that B is standard étale if we can choose such f, g so that there is an isomorphism of $A-$algebras $B \to (A[x]/(f))_g$. We then say that a ring homomorphism $\phi : A \to B$ is standard étale if it turns B into a standard étale $A-$algebra.

In fact this definition gives us all the morphisms we want since one way to define étale morphisms is being locally of finite presentation and locally standard étale. This is however slightly technical and uninituitive since it feels very restrictive to the case of mapping a space onto a hypersurface.

Going back to the implicit function to get a possibly easier to work with definition we have the following setup (taking for convenience our open subset on which the map is defined as the nonvanishing of some polynomial)

$$
\begin{tikzcd}
\{g \neq 0\} \cap \{(f_1 ... f_n) = 0\} \arrow[r, hook] \arrow[d, "\cong"] & \{(f_1 ... f_n) = 0\} \arrow[r, hook] \arrow[rr, bend left, "0"] & \mathbb{R}^{m+n} \arrow[r, "(f_1 ... f_n)"] \arrow[d, "\pi"'] & \mathbb{R}^n \\
U \arrow[rr, hook] & & \mathbb{R}^m &
\end{tikzcd}
$$

Whenever the jacobian of $f_1 ... f_n$ is non singular. And so dualising this by taking, for the sake of ease, regular functions we get the following

$$\begin{tikzcd}
A[x_{m+1}...x_{m+n}]_g/(f_1...f_n) & A[x_{m+1}...x_{m+n}]/(f_1...f_n) \arrow[l] & A[x_{m+1}...x_{m+n}] \arrow[l] & \mathbb{R}[x_1...x_n] \arrow[l, "(f_1...f_n)^*"'] \arrow[ll, bend right, "0"'] \\
C(U) \arrow[u, "\cong"'] & & A \cong \mathbb{R}[x_1..x_m] \arrow[ll] \arrow[u, hook]
\end{tikzcd}$$

And so we get the far more natural example of a local isomorphism, that of the ring map $A \to A[x_1...x_n]/(f_1...f_n)$, this is in our case a local isomrphism between the open sets $\{g \neq 0\}$ and U and so we take this to be our easier version of an étale map.

From [VAKI] we can define the following. For a morphism of schemes $\pi : X \to Y$, we say that this map is smooth of relative dimension n if there are open covers $\{U_i\}$ of X and $\{V_i\}$ of Y so that $\pi(U_i) \subset V_i \cong \operatorname{spec} B_i$ and the diagram

$$\begin{tikzcd}
U_i \arrow[r, "\cong"] \arrow[d, "\pi"'] & W \arrow[r, hook] \arrow[d, "\iota^*|_W"'] & \operatorname{spec} B_i[x_1...x_{n+r}]/(f_1...f_r) \arrow[dl, "\iota^*"] \\
V_i \arrow[r, "\cong"] & \operatorname{spec} B_i &
\end{tikzcd}$$

commutes where W is an open subscheme of $\operatorname{spec} B_i$ and $\det \frac{\partial f_i}{\partial x_j}$ is an invertible function on W. This is just the same as factoring through our local isomorphism from before except now instead of being locally an isomorphism it is locally an inclusion into a codimension n subspace. To extract our étale morphisms then we look at the case where $n = 0$

Definition 2.3.3. A morphism $f : X \to Y$ is étale if it is smooth of relative dimension 0

The natural example of an étale morphism is the map $k[t] \hookrightarrow k[t, t^{-1}]$ since as these are meant to represent open immersions we can see visually that the map $\mathbb{A}^1_k \setminus 0 \hookrightarrow \mathbb{A}^1_k$ is etale. In this case we have $k[t, t^{-1}]$ being a $k[t]-$algebra in the natural way so we just need to find $f, g \in k[t][x]$ so that this is $k[t][x]_g/(f)$. The natural choice is then to choose g to be the constant polynomial t and then for f' to be a unit we either need it to be a power of t or a constant, so we choose $f = x$, then we get the $k[t]-$alegebra $(k[t][x]/(x))_t = k[t, t^{-1}]$ so this map is infact étale

This gives us possibly the most natural topology on the category of schemes, we say that a family $\{\phi : U_\lambda \to S\}_{\lambda \in \Lambda}$ is a covering whenever the maps are étale and $\bigcup \phi(U_\lambda) = S$. These morphisms are naturally well behaved under base change so do form a topology. Note that this is weaker than the fppf topology, in that any étale covering is automatically fppf

This topology allows us to treat sheaves as geometric objects

Definition 2.3.4. A sheaf F over the étale site is called an Algebraic Space if

1. The diagonal $\Delta : F \to F \times F$ is representable by sheaves
2. There is a scheme T and etale surjection $T \to F$

Taking this up a step we can finally define things locally for stacks

Definition 2.3.5. A stack $\mathcal{F}$ over the fppf site is called a Deligne-Mumford stack if

1. The diagonal $\Delta : \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is representable by algebraic spaces
2. There is a scheme T and étale surjection $T \to \mathcal{F}$

2.4 The Smooth Topology

2.4.1 The differential world

Étale morphisms act as an algbraic analogue of open immersions. These are a very restrictive class of morphisms however. So as we do for many things in algebraic geometry we want to define our structures to be close to those in differential geometry.

In differential geometry we deal with manifolds, that is differential spaces that are locally modelled on $\mathbb{R}^n$ these however have a downside. For even slightly mean group actions we dont have quotients. For example if we take $\mathbb{R}^2$ and quotient by the $\mathbb{Z}/2\mathbb{Z}$ action $x \mapsto -x$ we see that when we take the quotient we end up with a cone which is going to fail to be smooth at the origin. To remedy this in the 50s mathematicians started to deal with "orbifolds" as to mean "orbit-manifold"[2]. These are spaces that are modelled on instead of $\mathbb{R}^n$, quotients of $\mathbb{R}^n$ by some linear action of a finite group, like the one mentioned earlier.

It turns out that these structures are well behaved if we treat them as a type of stack. However to get to that point we need to abstract away some of the differential structure. We start by generalising an orbifold to whats called a Lie-Groupoid.

Definition 2.4.1. A Lie groupoid is a groupoid $\mathcal{G}$ for whom $\operatorname{ob}\mathcal{G}$ and $\operatorname{mor}\mathcal{G}$ are manifolds, additionally the source and target maps $s, t : \operatorname{mor}\mathcal{G} \to \operatorname{ob}\mathcal{G}$ are submersions and the rest of the structure maps are smooth.

For example if we have a lie group G acting on a manifold M then the action groupoid $G \times M \rightrightarrows M$ is a lie groupoid since projection and multiplication are clearly submersions. We say that a Lie groupoid is proper if the source and target maps are proper, and étale if they are local diffeomorphisms. The quintessential example of a proper étale lie groupoid would then be the action groupoid of a finite group action, since the group is discrete the dimensions line up so everything is étale instead of a submersion and since it is finite the

[2]There was a suggestion to call them manifolded as they look like folded manifolds, this sadly didn't stick

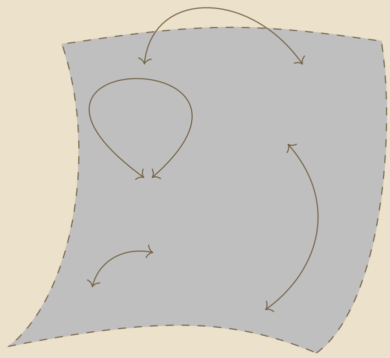

Figure 2.1: A visualisation of a Lie Groupoid

maps are clearly proper also. It turns out that this at least locally, the only thing a proper étale groupoid can be.

Theorem 2.4.2. *Let $\mathcal{G}$ be a proper étale Lie groupoid. Then for any point $x \in \operatorname{ob}\mathcal{G}$ there is an open neighborhood $U \subset \operatorname{ob}\mathcal{G}$ so that the restriction $\mathcal{G}|_U$ is isomorphic to the action groupoid $\Lambda \times U \rightrightarrows U$, where Λ is a finite group. Moreover we can take U to be an open ball in $\mathbb{R}^n$ centered at 0 and the action of Λ to be linear.*

Proof. See [LERM] □

This means that we can treat orbifolds as a special case of this new Lie groupoid structure. Coming back to stacks, one of the important things that stacks allow us to do is to take quotients. In differential geometry orbifolds allow us to take quotients, and so we would like to be able to treat these as the same thing.

We can define stacks over any category we want as long as we give it some Grothendieck topology. And so we can do so on the category of differentiable manifolds.

Definition 2.4.3. The site of smooth manifolds is just the category of manifolds **Mfd** along with a Grothendieck topology where $\phi : U_i \to M$ cover M if $\bigcup \phi_i(U_i) = M$

An important example of a stack over this category is given by the following. For a lie groupoid $\mathcal{G}$ we can construct the classifying stack $B\mathcal{G}$ of principal G bundles. We will see later that BG is generally just the stack $[*/G]$. In terms of fibered categories (see Appendix B) we define the following

Definition 2.4.4. For a Lie groupoid $\mathcal{G}$ we can define the classifying stack $B\mathcal{G}$ as follows

1. The objects of $B\mathcal{G}$ are principal $\mathcal{G}$ bundles

2. The morphisms are then just $\mathcal{G}$ equivariant maps

This construction can be made functorial if one wants

Since this definition is functiorial and well behaved, If you want to treat stacks as your world of spaces then one could say that an orbifold is just its classifying stack and so orbifolds are precicely just a special type of stack. In fact they are a large proportion of all stacks

We want to, as is usual, consider stacks that are locally well behaved. Ie in this case those who are representable by manifolds. Akin to our previous definition requiring a surjective map we can take a well behaved covering of our stack in order to encapsulate this property.

Definition 2.4.5. An atlas of a stack is a manifold X and a map $X \to D$ so that for any manifold map $M \to D$ the fiber product $M \times_D X$ is a manifold and the projection $M \times_D X \to M$ is a surjective submersion

Definition 2.4.6. A stack with an atlas is called a geometric stack, these are the differentiable analogue of our algebraic stacks

Luckily as usual the theory of differentiable strucures is more restricitve than that of algebraic strucutres so we can actually categorise all geometric stacks they are all just Lie groupoids

Theorem 2.4.7. *Any geometric stack $\mathcal{D}$ is isomorphic to $B\mathcal{G}$ for some Lie groupoid $\mathcal{G}$*

Proof. See [LERM] $\square$

And so we can see that very neatly the theory of stacks contains the theory of orbifolds and so the theory of quotients. If then we want to treat quotients nicely in our algebraic context we would like to deal with some analogue of these geometric stacks. That is stacks that come from smooth morphisms.

2.4.2 The algebraic world

This treatment in differential geometry gives us reason to trust that smooth stacks encode the correct information to treat the theory of orbifolds and quotients. So we mimic this directly in the algebra.

Recall that we say that a morphism is smooth if locally, it is just a morphism $A \to A[x_1...x_n]/(f_1...f_s)$ where

$$\det\left(\frac{\partial f_i}{\partial x_j}\right)$$

Is a unit. We can then use this to define the most common type of stack

Definition 2.4.8. A stack $\mathcal{F}$ over the fppf site is called an Algebraic stack if

1. The diagonal $\Delta : \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is representable by algebraic spaces

2. There is a scheme T and smooth surjection $T \to \mathcal{F}$

Note that since étale maps are smooth we see immediatly that we have a series of weakenings

$$\text{Schemes} \subset \text{Alegbraic Spaces} \subset \text{Deligne-Mumford Stacks} \subset \text{Algebraic Stacks}$$

One nice thing about these algebraic stacks is that there is a simple approach to proving that a stack is algebraic. One can just find a nice map to another known algebraic stack and if this map is representable then we must have a stack.

Proposition 2.4.9. *[STAC]For an fppf scheme S and morphism of S-stacks $\mathcal{X} \to \mathcal{Y}$. If*

1. *$\mathcal{X} \to \mathcal{Y}$ is representable by algebraic spaces*
2. *$\mathcal{Y}$ is an algebraic stack over S*

Then $\mathcal{X}$ is an algebraic stack

Proof. This proof is a purely technical result that fiber products of a stack and a representable stack is representable and that smooth morphisms are stable under these fiber products so an atlas of $\mathcal{Y}$ pulls back through the fiber product to an atlas on $\mathcal{X}$ making it algebraic. For details see Stacks Project 94.15.4 $\square$

Chapter 3

Groups

3.1 1-groups

In order to find out what a group stack is we must first know what a group is.

Definition 3.1.1 (*Group in the category of sets*)**.** A group is a pair $(G, \circ)$ for G a nonempty set and $- \circ - : G \times G \to G$ satisfies the commutative diagram

$$\begin{tikzcd} G \times G \times G \arrow[r, "(id,\circ)"] \arrow[d, "(\circ,id)"'] & G \times G \arrow[d, "\circ"] \\ G \times G \arrow[r, "\circ"'] & G \end{tikzcd}$$

And for any $g \in G$ the maps $g \circ -, - \circ g$ are isomorphisms

Corollary 3.1.2. *This definition recovers the standard definition of a group*

Proof.

1. The commutative square gives us exactly associativity
2. For some $g \in G$ by bijectivity we must have unique left and right identities $e_g \circ g = g \circ f_g = g$, thanks then to associativity $e_g \circ e_g \circ g = e_g \circ g = g$ so $e_g \circ e_g = e_g$, then for any h by $e_g \circ -$ being an isomorphism we can find some ι so that $e_g \circ \iota = h$. This means that $h = e_g \circ \iota = e_g \circ e_g \circ \iota = e_g \circ h$. So e_g is universally a left identity. Equally f_g is universally a right identity and $e_g = e_g \circ f_g = f_g$ so we have an identity
3. Since for any $g \in G$, $g \circ -$ is bijective there must be some g^{-1} so that $g \circ g^{-1}$ is the identity this is too a left inverse since we have a left inverse ℓ and $\ell = \ell \circ g \circ g^{-1} = g^{-1}$ so we have inverses

$\square$

This definition then allows us to define what it means to be a group object in any diagram category $[\mathcal{C}, \mathbf{Set}]$ since any products occur over in **Set**, so we can say that a group object in $[\mathcal{C}, \mathbf{Set}]$ is a functor who takes values in the group objects of **Set**. This is really enough for our needs since our stacks are functors but we can generalise even further. For any locally small category $\mathcal{C}$ we have a continuous (thus product preserving) embedding $よ : \mathcal{C} \hookrightarrow [\mathcal{C}, \mathbf{Set}]$ and so we can define for any locally small category group objects as those objects who have hom groups.

3.2 The multiplicitive group

An important example of a group object is that of the multiplicitive group $\mathbb{G}_m$. If we have a k-scheme X then one thing we would want to find is the group of units $\mathcal{O}_X(X)^\times$. One can easily compute that points of $\mathcal{O}_X(X)^\times$ are just k-algebra maps $k[x, x^{-1}] \to \mathcal{O}_X(X)^\times$ and so this functor $X \mapsto \mathcal{O}_X(X)^\times$ is a sheaf and in fact a scheme corresponding to $\hom(-, k[x, x^{-1}])$. Since each $\mathcal{O}_X(X)^\times$ is a group this is thus very naturally a group scheme

3.3 2-groups

We want to extend this definition then to group structures in higher categories. When we do so we no longer have the luxury of equality[1] and so we must only look up to coherent isomorphisms.

Definition 3.3.1. A group groupoid or 2-group is a groupoid $\mathcal{G}$ along with a binary operation $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and an invertable 2-morphism $\alpha : (- \circ -) \circ - \implies - \circ (- \circ -)$ such that

1. For every $g \in G$ the functors $g \circ -, - \circ g$ are equivalences, ie they have inverses up to natural isomorphism
2. The diagram (Which we will call this the coherence cube)

$$\begin{tikzcd}
 & & G \times G \times G \times G \arrow[dll] \arrow[d] \arrow[drr] & & \\
G \times G \times G \arrow[d] \arrow[drr] \arrow[rrrr, Rightarrow, bend right] & & G \times G \times G \arrow[dll] \arrow[drr] & & G \times G \times G \arrow[dll] \arrow[d] \\
G \times G \arrow[drr] & & G \times G \arrow[d] & & G \times G \arrow[dll] \arrow[llll, Rightarrow, bend right] \\
 & & G & &
\end{tikzcd}$$

[1]To some the fact we had equality in the first place would be more worrying than our loss of it, see: `https://ncatlab.org/nlab/show/evil`

2-commutes, that is every composition of 1-morphisms is the same up to composition of 2-morphisms, and the 2-morphisms commute

For the sake of intuition it is worth describing exactly what that diagram does, if we chase some tuple x, y, z, w we get the following

$$\begin{tikzcd}
 & & x,y,z,w \arrow[dll] \arrow[d] \arrow[drr] & & \\
xy,z,w \arrow[d] \arrow[drr] & & x,yz,w \arrow[dll] \arrow[drr] & & x,y,zw \arrow[dll] \arrow[d] \\
(xy)z,w \cong_\alpha x(yz),w \arrow[drr] & & xy,zw \arrow[d] & & x,(yz)w \cong_\alpha x,y(zw) \arrow[dll] \\
 & & \heartsuit & &
\end{tikzcd}$$

Where $\heartsuit$ denotes the expression

$$((xy)z)w \cong_\alpha (x(yz))w \cong_\alpha (xy)(zw) \cong_\alpha x(yz)w \cong_\alpha x(y(zw))$$

So we see that this cube is just a really complicated way to show that when we apply assoicativity with α we get the same isomorphism nomatter which order we associate in.

This definition is fine but as is common when doing category theory you can describe an object by its properties but as a mathematician you want to know at some point whats going on inside. In order to do so we want to appeal to some structure internal to the category. For this we need to define a more standard object.

Definition 3.3.2 (*Monoidal Category*)**.** A monoidal category is a category $\mathcal{C}$ along with

1. A functor $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
2. An object I called the unit
3. Natural isomorphisms $\lambda : I \otimes - \to \text{id}$, $\rho : - \otimes_{\to} \text{id}$ called left and right unitors respectively and an associator $\alpha : (- \otimes -) \otimes - \to - \otimes (- \otimes -)$

Such that these natural isomorpisms are coherent, in the sense that for any $A, B, C, D \in \mathcal{C}$

the diagrams

$$\begin{tikzcd}
 & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A \otimes B,C,D}"] & \\
A \otimes (B \otimes (C \otimes D)) \arrow[ur, "\alpha_{A,B,C \otimes D}"] \arrow[d, "1_A \otimes \alpha_{B,C,D}"'] & & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes) C \otimes D) \arrow[rr, "\alpha_{A,B \otimes C,D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha_{A,B,C} \otimes 1_D"']
\end{tikzcd}$$

And

$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "1_A \otimes \lambda_B"'] & & (A \otimes I) \otimes B \arrow[dl, "\rho_A \otimes 1_B"] \\
 & A \otimes B &
\end{tikzcd}$$

commute for every A, B, C, D

These diagrams are a little confusing but all that it is really saying is that when you associate it does as you would expect, ie any way you associate some expression ends up with the same overall morphism.

Proposition 3.3.3. *A 2-group is precicely a monoidal category $(\mathcal{C}, \otimes)$ such that $\mathcal{C}$ is a groupoid and the monoidal structure is invertable up to isomorphism, ie for any $A \in \mathcal{C}$ there exists $A^{-1} \in \mathcal{C}$ so that $A \otimes A^{-1} \cong I \cong A^{-1} \otimes A$*

Proof. If we have a 2-group there is a monoidal structure given by the map $G \times G \to G$. By the 2-commuting of the coherence cube we can see that there is a canonical natural isomorphism between any way of associating and so the pentagonal diagram is satisfied. To find an indentity we can mimic the proof for 1-groups to conclude that up to isomorphism there is a unique identity element 1, it is then clear that we have inverses up to isomorphism since it is again the same proof but all up to isomorphism. The existence of an identity allows us to check the triangle which again follows from the coherence cube after applying the equivalence $A^{-1} \cdot -$ If we we're to restrict ourselves to strict 2-groups then we could conclude that this is a groupoid when only assuming that it is a category (see [BROW]), but as it's unclear wether this is possible in the non strict case

If instead we start with a monoidal groupoid with inverses then the requirements of a 2-group are followed immediately since the inverses make the maps equivalences and the diagram follows from the pentagon identity. $\square$

3.4 Fundamental n-groups

This section uses the language of $\infty-$categories, for a primer see Appendix C. To make this seem an almost obvious way to define a 2-group, recall that a group can be seen as a groupoid with a single object. One can then see this as an $\infty-$groupoid and thus as a space by taking the geometric realisation of this simplicial set with simplices in degrees $0, 1, 2$ we see that we recover the original group as the fundamental group of this space, in fact this is precicely the a construction for $K(G, 1)$ the standard first Eilenburg-Maclane space of G.

By this interpretation a group is just the fundamental group of some ∞-groupoid/space and so the data of the group is perhaps better put in just the space itself. That is the data of a group is within the data of a connected pointed space. If then we had a space with higher dimensional simplices we would have to consider the way in which these paths are homotopic. This means we would have a groupoid of loops with homotopies between them along with a monoidal structure given by concatenation. This is just a 2-group! This construction is called the fundamental 2-group which we can see now is just the obvious thing one would do to track loops when there is homotopies involved. In fact one could use this to define n-groups as the fundamental n-group of some space where we take the $n-1$ groupoid of loops with homotopies and then have a monoidal structure of concatenation. This then allows us to define ∞-groups very simply, they are just the loop space of some pointed, connected space[2] with a monoidal structure given by concatenation. This is the more general way to picture an n-group, as just the obvious construction one would make given the data of loops with some level of homostopy

[2]here it may be better to say ∞-groupoid instead of space, it's however very common to equivocate the two

Chapter 4

Quotient Stacks

4.1 Quotients

Suppose that we have an object X and a group object G acting on X. Say these objects are sets, then the quotient object X/G the set of all orbits of objects in X. In other words we take the set X but we add the condition that for all g, x $x = gx$. This is the same as saying that from the perspective of X/G the two maps $(g, x) \mapsto x$ and $(g, x) \mapsto gx$ are the same. That is we just take the colimit of the diagram

$$G \times X \rightrightarrows X$$

Taking this up a step to groupoids recall that a 2-group action has additional structure of an associator $(G \times G) \times X \to G \times (G \times X)$ and so when forming the quoient we need to care about this structure too, that is we want $(gh)x \simeq g(hx) = gx = x$. No matter which order we associate we still have $ghx = x$. This means we must add this second condition to our diagram meaning we actually take the colimit of the diagram

$$G \times G \times X \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; G \times X \rightrightarrows X$$

In this essay we need only technically look up to 2-categories is is however often useful to appeal to infinity categories (see Appendix C). In this case the diagram extends to taking the colimit of the whole simplicial diagram

$$\ldots \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; G \times G \times X \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; G \times X \rightrightarrows X$$

This just being the geometric realisation of this diagram. This interpretation makes our lives occasionally easier and we can see that it gives back our small diagrams when we truncate since the higher terms only give information about the higher dimensional cells and in our cases we ignore them. This interpretation allows us to appeal to strong results in infinity categories and then just truncating to our 2-categorical world.

One of the nice things then about Algebraic spaces and Stacks is using these definitions we can get more types of quotient than we could before by just computing these colimits since we just need to compute in **Set** or **Gpd** which are very nice to work in categories.

Proposition 4.1.1 (Quotient Stack)**.** *For X a scheme over S and G an affine smooth group scheme over S with an action on X. We define the quotient stack $[X/G]$ is the functor $(Sch/S)_{fppf} \to \mathbf{Gpd}$ where*

1. *For a scheme T $[X/G](T)$ is the groupoid of spans $T \leftarrow P \to X$ where $P \to T$ is a principal G bundle and $P \to X$ is an equivariant map*

This is equivalently by Appendix B the fibered category

1. *For a scheme T an object over T is a principle G bundle $P \to T$ together with an equivariant map $P \to X$*
2. *a morphism from $P \to T$ to $P' \to T'$ is a bundle map, that is compatible with the equivariant maps $P \to X, P' \to X$*

Often one takes this as the definition of a quotient stack. From [KHAN] we have that both this and the (∞-)colimit definition are equvalent

Proof. Suppose we have the quotient stack $[X/G]$ defined by the colimit. By the Yoneda Lemma, the data of $[U/G](T)$ is just the data of the morphisms $T \to [U/G]$. For any such morphism we can get a unique cartesian square

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\pi"] & U \arrow[d, "p"] \\ T \arrow[r] & {[U/G]} \end{tikzcd}$$

Where Y, U are principal $G-$bundles which is then precicely the data of the definition

If instead we start with the principle $G-$bundle $T \leftarrow Y \to U$. By a theorem of Lurie[1] we may construct our morphism $T \to [U/G]$ as $T \cong [Y/G] \to [U/G]$ □

The question one must ask at this point is wether this quotient stack can be treated geometrically. As we've discussed one should expect that reasonably okay quotients give us algebraic stacks

Theorem 4.1.2. *[KHAN] Let G be a smooth group algebraic space over a scheme S, for a stack U with a G action, if U is algebraic then so is $[U/G]$*

[1]Theorem 4.9 in [KHAN]

Proof. We obtain an atlas of $[U/G]$ by simply factoring the atlas of U through the smooth surjection $U \to [U/G]$

The representability of the diagonal is reliant on some technical results I haven't proven, see [KHAN] for details. $\square$

4.2 Groups

One common situation in algebraic geometry is having multiple group schemes at acting on one another. In this case you might want to take the quotient of these and hope that it's still a group. In some cases this is possible

Proposition 4.2.1. *For a ring R and group schemes over $\operatorname{spec} R$ G, H flat and of finite presentation with a normal containment $H \trianglelefteq G$, then there exists an fppf sheaf G/H so that we have a short exact sequence*

$$1 \to H \to G \to G/H \to 1$$

Where G/H exists as an algebraic space

Proof. This is constructed as the sheafification of the presheaf $X \to G(X)/H(X)$ $\square$

This is a nice theorem that allows us to treat reasonably well behaved quotients, but what about the bad ones? To treat those we would have to use stacks

Theorem 4.2.2. *For a group scheme G acting on a group scheme X if the action commutes, in the sense that $xy \cdot ab = (x \cdot a)(y \cdot b)$ and the quotient $[X/G]$ exists as an algebraic stack, then $[X/G]$ is a group stack in a natural way*

Proof. By [BEJF] we know that $[X \times X/G \times G] \cong [X/G] \times [X/G]$ and so we have the diagram

$$\begin{tikzcd}
G \times G \times X \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G \times X \arrow[r, shift left=1] \arrow[r, shift right=1] & X \arrow[r, dashed] & {[X/G]} \\
\begin{matrix}(G \times G)\times \\ (G \times G)\times \\ (X \times X)\end{matrix} \arrow[u] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \begin{matrix}(G \times G)\times \\ (X \times X)\end{matrix} \arrow[u] \arrow[r, shift left=1] \arrow[r, shift right=1] & X \times X \arrow[u] \arrow[r, dashed] & {[X/G] \times [X/G]} \arrow[u, dotted]
\end{tikzcd}$$

Which by the assumption of commutativity this diagram commutes and the morphism m exists up to unique natural isomorphism. Associativity holds strictly within the diagram and so when we choose two different morphisms for the two different associations $[X/G] \times [X/G] \times [X/G] \to [X/G]$, they make the same diagram commute and so there is a unique natural transformation between them, this is then the associator. We can see that the associator cube 2-commutes by uniqueness of this choice of associator and so we have a group structure on $[X/G]$ □

4.3 Classifying Stacks

One important example of a quotient stack is the classifying stack. If we have some group scheme G over some scheme $\operatorname{spec} k$ one natural datum to consider is the stack of principal G-bundles. This by our equivalence of definitions from earlier is the stack $[\operatorname{spec} k/G] := BG$ and we can see that [HOER] if this G is an abelian group scheme over $\operatorname{spec} k$ then this is in fact a group stack, and is for many cases the most natural way to categorify our group scheme to a stack instead of the natural approach of just turning a set into its natural discrete groupoid

4.4 Abelian Group Stacks

The most important type of group stack is that of an abelian group stack. These are the main type of group stack that are studied. This type of stack is commonly reffered to as a "Picard Stack" however we refrain from using this terminology as it would makes the naming in Chapter 5 a bit of a pain, for this reason they will be referred to as abelian group stacks.

To be clear what is meant by abelian in this case is that the functor takes values in *symmetric/abelian* 2-groups. That being monoidal categories with inverses and an additional natural transformation informing how to commute two objects

Definition 4.4.1. A symetric monoidal category is a monoidal category with an additional commutator natural isomorphism $s_{A,B} : A \otimes B \to B \otimes A$ so that the following diagrams commute

$$\begin{tikzcd}
A \otimes I \arrow[rr, "s_{AI}"'] \arrow[dr, "\rho_A"] & & I \otimes A \arrow[dl, "\lambda_A"'] \\
& A &
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes B \arrow[rr, equal, "id_{A\otimes B}"] \arrow[dr, "s_{AB}"'] & & A \otimes B \\
& B \otimes A \arrow[ur, "s_{BA}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (A \otimes B) \otimes C \arrow[r, "s_{AB}\otimes id_C"'] \arrow[dl, "\alpha_{ABC}"] & (B \otimes A) \otimes C \arrow[dr, "\alpha_{BAC}"'] & \\
A \otimes (B \otimes C) \arrow[dr, "s_{A,B\otimes C}"] & & & B \otimes (A \otimes C) \arrow[dl, "id_B \otimes s_{AC}"'] \\
& (B \otimes C) \otimes A \arrow[r, "\alpha_{BCA}"] & B \otimes (C \otimes A) &
\end{tikzcd}$$

If then our stack takes values in this subcategory of abelian 2-groups we have an abelian group stack.

Thanks to this fact about quotients we have a very natural way to generate abelian group stacks, from a morphism of group schemes $H \to G$ we can construct the quotient $[G/H]$ as an abelian group stack. In fact this gives us often all abelian group stacks in the following sense[SGA4].

Such a pair of schemes $H \to G$ can be seen as a complex of sheaves concentrated in degrees $\{-1, 0\}$, call this category $C^{-1,0}$ for such a complex we can define a prestack as follows

Definition 4.4.2. For a complex $(d : K_{-1} \to K_0) = K \in c^{-1,0}$ we can define a prestack $\mathrm{pch}(K)$ as follows, for an object U we have

1. $\mathrm{ob}(\mathrm{pch}(K)(U)) = K_{-1}(U)$
2. For $x, y \in K_{-1}(U)$, $\hom_{\mathrm{pch}(K)(U)}(x, y) = \{p \in K_{-1} | dp = x - y\}$
3. We compose $f \circ g = f + g$, note that this is well defined as $d(f + g) = df + dg = (x - y) + (y - z) = x - z$

In the case that these K_0, K_1 are group objects we can inherit a group prestack structure on pch, we then define ch as the functor that takes K to the stackification of $\mathrm{pch}(K)$. See Appendix A for an explanation of sheafification, stackification is similar but more complicated. This allows us to classify all abelian group stacks.

Theorem 4.4.3. *[SGA4]The functor* ch *induces an equivalence between the derived category of complexes of abelian sheaves over any site S with representatives concentrated in degrees $-1, 0$ and the category of abelian group stacks over this site.*

[HOER]In the special case of sheaves over $\operatorname{spec} k$ *this equivalence is from the derived category of abelian group schemes over k and the functor* ch *corresponds to the quotient map*

$$(H \to G) \mapsto [G/H]$$

So a critical takeaway is that the study of abelian group stacks is equivalent to the study of abelian sheaves[2]

[2]This is possibly similar to the relation between 2-groups and crossed modules

Chapter 5

The Picard Stack

5.1 Line Bundles

In geometry one of the natural objects of study is that of vector bundles, they come with a natural structure given by the tensor of vector bundles.

Definition 5.1.1 (*Vector Bundles)*. We define a vector bundle of rank n over a scheme, or more generally a locally ringed space, to be a sheaf of $\mathcal{O}_X$ modules $\mathcal{F}$ that is locally free of rank n, ie for each point x there is some neighborhood U of x so that

$$\mathcal{F}(U) \cong \mathcal{O}_X(U)^{\oplus n}$$

To then define our tensor product of sheaves we can simply sheafify the presheaf of point-wise tensor products.

Definition 5.1.2 (*Tensor of Sheaves)*. For a pair of sheaves of $\mathcal{O}_X-$modules $\mathcal{F}, \mathcal{G}$ we can define the tensor product presheaf $(\mathcal{F} \otimes_{pre,\mathcal{O}_X} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. We can then define the tensor product

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = \mathcal{F} \widetilde{\otimes_{pre,\mathcal{O}_X}} \mathcal{G}$$

it is clear from the definitions that $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{G}_x$

From this we can see that the tensor of a rank m vector bundle and a rank n vector bundle is a rank mn vector bundle. One thing we see here is that if $m = n = 1, mn = 1$, ie the collection of rank 1 vector bundles is closed under the operation of tensoring.

Definition 5.1.3 (*Line Bundles I)*. We say that a sheaf is a line bundle if it is a vector bundle of rank 1

We will, somewhat suggestively, refer to the collection of line bundles on X as $\mathrm{Pic}(X)$. For the sake of intuition, lets look at the most famous line bundle, the Möbius band.

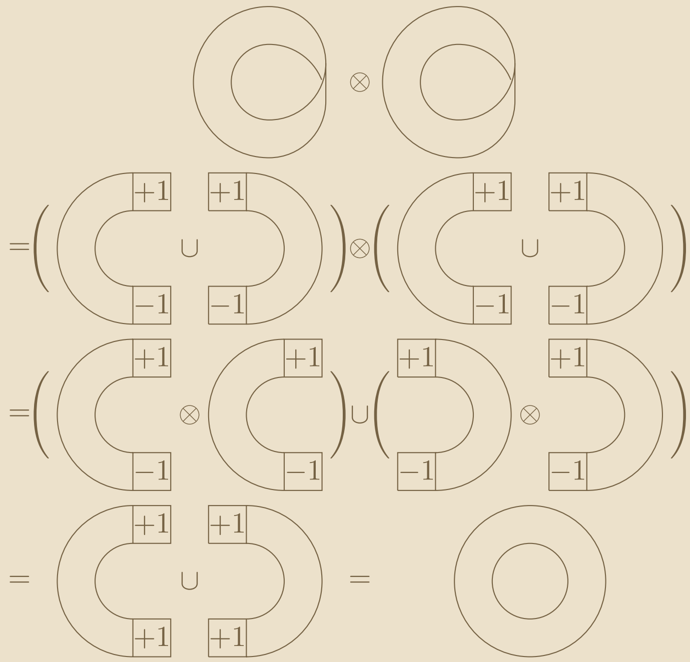

Figure 5.1: The tensor product of the Möbius band with itself

In this case, for the sake of visualisation, we will retreat to the standard definition of a vector bundle, that of local trivialisations with gluing maps $U_i \cap U_j \to GL(n)$, in this case our tensor product just gives you the tensor of the vector spaces where we take some common trivialisation and then the gluing maps are just the tensor of the original gluing maps.

Example 5.1.4. Taking the Möbius band M we can split it up into local trivialisations and by just computing in Fig. 5.1 we see that $M \otimes_{S^1} M = S^1 \times \mathbb{R}$. In fact in the case of S^1, there are only 2 line bundles, namely the Möbius band and the trivial line bundles[ELEN], and we can see now that they form a group structure, $\mathrm{Pic}(S^1) \cong \mathbb{Z}/2\mathbb{Z}$

It is in fact the case that this $\mathrm{Pic}(X)$ always forms a group

Theorem 5.1.5. *For a sheaf $\mathcal{L}$ on a locally ringed space $(X, \mathcal{O}_X)$, the following are equivalent*

1. *The evaluation map $\mathcal{L} \otimes \hom(\mathcal{L}, \mathcal{O}_X) \to \mathcal{O}_X$ is an isomorphism, ie the dual sheaf of $\mathcal{L}$ is an inverse.*
2. *$\mathcal{L}$ is invertible, ie there exists a sheaf $\mathcal{N}$ so that $\mathcal{L} \otimes_X \mathcal{N} \cong \mathcal{O}_X$*
3. *$\mathcal{L}$ is a line bundle (locally free sheaf of rank 1)*

Proof. [YUAN]Clearly $1 \implies 2$. To show that $3 \implies 1$ we can just compute, it is sufficient to show that this is an isomorphism on stalks, so sufficient to restrict to a local trivialisation, at this trivialisation our evaluation map becomes $\mathcal{O}_U \otimes_U \mathcal{O}_U \to \mathcal{O}_U$ mapping

$s \otimes t \mapsto st$, this is clearly surjective and injective since $st = 0 \implies s \otimes t = st1 \otimes 1 = 0$. We're then just left with proving that $2 \implies 3$

[STAC]Suppose that our sheaf is invertable, so there is an isomorphism $\varphi : \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N} \to \mathcal{O}_X$. For any point x, take an open neighborhood of x and sections $s_i \in \mathcal{L}(U), t_i \in \mathcal{N}(U)$ so that $\varphi(\sum_i^n s_i \otimes t_i) = 1$. Consider then the isomorphisms

$$\begin{array}{ccccc} & & s \otimes s' \otimes t & \longmapsto & \sum \varphi(s \otimes t_i) s_i \\ \mathcal{L}(U) & \longrightarrow & \mathcal{L}(U) \otimes_{\mathcal{O}_U} \mathcal{L}(U) \otimes_{\mathcal{O}_U} \mathcal{N}(U) \otimes_{\mathcal{O}_U} & \longrightarrow & \mathcal{L}(U) \otimes_{\mathcal{O}_U} \\ s & \longmapsto & \sum s_i \otimes s \otimes t_i & & \end{array}$$

This isomorphism factors through $\mathcal{O}_U^{\oplus n}$ by mapping

$$s \mapsto (\varphi(s \otimes t_1), ..., \varphi(s \otimes t_n)) \mapsto \varphi(s \otimes t_1)s_1 + ... + \varphi(s \otimes t_n)s_n$$

and so we have a split exact sequence $0 \to \mathcal{L}(U) \hookrightarrow \mathcal{O}_U^{\oplus n} \to \mathcal{M} \to 0$ so $\mathcal{L}$ is a summand of a finite free $\mathcal{O}_U$-module. This means that at each stalk $\mathcal{L}_x$ we have a finitely genereted projective module over the local ring $\mathcal{O}_x$ and so each stalk is free and so the sheaf itself is locally free, clearly it must be rank 1 as for vector bundles $\text{rank}(V \otimes U) = \text{rank}(V) \cdot \text{rank}(U)$ □

This leads us to an alternate definition of line bundles that is ocasionally more useful than considering vector bundles

Definition 5.1.6 (Line Bundles II)**.** A line bundle over a (locally) ringed space X is a sheaf, invertible with respect to the tensor product

5.2 Čech cohomology

There is another very useful formulation of this group $\text{Pic}(X)$, it can be defined homologically.

Definition 5.2.1. For a space X we can define for any sheaf of abelian groups $\mathcal{A}$ the sheaf cohomology groups $H^i(X, \mathcal{A})$ by taking the left exact functor $\Gamma(X, -) : \mathbf{Sh(X)} \to \mathbf{Ab}$, that takes a sheaf to its group of global sections, and computing its right derived functors $\mathrm{H}^i(X, \mathcal{A}) := \mathrm{R}^i\Gamma(X, \mathcal{A})$

Within then this framework we can write $\text{Pic}(X)$ quite simply

Theorem 5.2.2. *The picard group* $\text{Pic}(X)$ *on a ringed space is isomorphic to the sheaf cohomology group* $H^1(X, \mathcal{O}_X^\times)$ *where* $\mathcal{O}_X^\times$ *is the sheaf of units of the ring* $\mathcal{O}_X$

In order to prove this theorem we must pass through a gadget called Čech cohomology. This is a cohomology theory designed to look at the cohomology of a sheaf but with respect to some open cover, we can see immediatly why this may be useful as it allows us to look only where our line bundles are trivial and compute cohomology there instead of having to deal with the whole space as is. To do so we first define the Čech complex relative to some open cover $\mathcal{U}$

Definition 5.2.3. For an open cover of X $\mathcal{U} = \{U_i\}_{i \in I}$, the Čech complex $C^i(\mathcal{U}, \mathcal{F})$ is a complex of sheaves over X defined as follows. For a sequence of indeces $i_0 < ... < i_j$ we define the sheaf

$$C^{i_0,\dots,i_j}(\mathcal{U}, \mathcal{F})(V) = \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_j} \cap V)$$

Note that this is infact a sheaf of abelian groups as it is just the pushforward of $\mathcal{F}|_{U_{i_0} \cap \cdots \cap U_{i_j}}$ under the inclusion map $U_{i_0} \cap \cdots \cap U_{i_j} \hookrightarrow X$.

Definition 5.2.4. We then define the jth sheaf of Čech cocycles then as

$$C^j(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \dots < i_j} C^{i_0,\dots,i_j}(\mathcal{U}, \mathcal{F})$$

This determines a sheaf of abelian groups as it is just the product of such sheaves.

We have determined a sequence of sheaves and so in order to do cohomology to these objects we need to determine our differentials. We define this termwise in the product

Definition 5.2.5. The Čech cochain complex is a complex

$$C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F}) \to \dots$$

Where the differential is given by for an element $s \in C^j(\mathcal{U}, \mathcal{F})(V)$

$$(\delta s)_{i_0 \dots i_{j+1}} = \sum_{n=0}^{j+1} (-1)^n s_{i_0 \dots \hat{i_n} \dots i_{j+1}}|_{U_{i_0} \cap \dots \cap \hat{U_{i_n}} \cap \dots U_{i_j} \cap V}$$

It is a straightforward computation to check that this is a well defined map of sheaves and that $\delta^2 = 0$

We can now see precicely in what sense Čech cohomology is considered as an approximation to sheaf cohomology, to go from this cochain complex to the homology then we just do as we would for sheaf cohomology, we take global sections and compute homology there. The reason we passed to this complex of sheaves is so that we can find a direct comparison map.

Definition 5.2.6. We define the Čech cohomology groups of some sheaf relative to some open cover $\mathcal{U}$ as

$$\check{\mathrm{H}}^i(\mathcal{U}, \mathcal{F}) := \mathrm{H}^i(\Gamma(X, C^\bullet(\mathcal{U}, \mathcal{F})))$$

To finally define a version without reference to an open cover we just take the limit over finer and finer covers

Definition 5.2.7. The Čech cohomology group then without a specified open cover is simply the direct limit

$$\check{\mathrm{H}}(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^i(\mathcal{U}, \mathcal{F})$$

Where the open covers are ordered by refinement

To compare this $\check{\mathrm{H}}(X, \mathcal{F})$ with our standard $\mathrm{H}(X, \mathcal{F})$ we first compare with how we define $\mathrm{H}(X, \mathcal{F})$ to begin with, that is we take an injective resolution of our sheaf $\mathcal{F}$

$$\mathcal{I}(\mathcal{F}) : \quad 0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to ...$$

Then take global sections then take homology, so the point at which we want compare these two constructions is at the beginning since that is when theyre most barebones. Note that when we look at $C^j(\mathcal{U}, \mathcal{F})$ we are looking at the intersection of $j+1$ sets, so C^0 will talk about single sets and C^{-1} talks about no sets, so C^{-1} is simply $\mathcal{F}$ so we have the two cochains

$$\begin{tikzcd}
\mathcal{I}(\mathcal{F}) \colon & 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{I}^0 \arrow[r] & \mathcal{I}^1 \arrow[r] & \mathcal{I}^2 \arrow[r] & \dots \\
C(\mathcal{U}, \mathcal{F}) \colon & 0 \arrow[r] & \mathcal{F} \arrow[r] & C^0(\mathcal{U}, \mathcal{F}) \arrow[r] & C^1(\mathcal{U}, \mathcal{F}) \arrow[r] & C^2(\mathcal{U}, \mathcal{F}) \arrow[r] & \dots
\end{tikzcd}$$

And so by the definition of injective this inductively induces maps, unique up to homotopy

$$\begin{tikzcd}
\mathcal{I}(\mathcal{F}) \colon & 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{I}^0 \arrow[r] & \mathcal{I}^1 \arrow[r] & \mathcal{I}^2 \arrow[r] & \dots \\
C(\mathcal{U}, \mathcal{F}) \colon & 0 \arrow[r] & \mathcal{F} \arrow[r] & C^0(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[u] & C^1(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[u] & C^2(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[u] & \dots
\end{tikzcd}$$

And so we get well defined induced maps in homology $\check{\mathrm{H}}^i(\mathcal{U}, \mathcal{F}) \to \mathrm{H}^i(X, \mathcal{F})$ which then of course extend to a map of the direct limit $\check{\mathrm{H}}^i(X, \mathcal{F}) \to \mathrm{H}^i(X, \mathcal{F})$. There are then theorems about this comparison map, for example one can find that when the cech cohomology vanishes with some cover this is an isomorphism (Cartans Theorem), If the sheaf cohomoloy vanishes on some collection of subspaces then the relative version of this map is an isomorphism (Leray's Theorem), this map is injective for $i = 2$ and more. The one that we care about however is the followng

Theorem 5.2.8. *For a topological space X and sheaf $\mathcal{F}$ on X then the map*

$$\check{\mathrm{H}}^i(X, \mathcal{F}) \to H^i(X, \mathcal{F})$$

Is an isomorphism for $i = 0, 1$

Proof. At 0 this is just the identity map of global sections so theres nothing to check.

At 1 then we can first embed $\mathcal{F}$ into a flasque sheaf $\mathcal{G}$ to get the following exact sequences for some $D^\bullet$ and $\mathcal{R}$

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{R} \to 0$$

$$0 \to C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{G}) \to D^\bullet(\mathcal{U}) \to 0$$

Taking homology of this and using that the Čech and sheaf cohomology of flasque sheaves vanishes we get the exact sequences

$$0 \to H^0(X, \mathcal{F}) \to \mathrm{H}^0(X, \mathcal{G}) \to \mathrm{H}^0(X, \mathcal{R}) \to \mathrm{H}^1(X, \mathcal{F}) \to 0$$

$$0 \to \check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^0(\mathcal{U}, \mathcal{G}) \to \mathrm{H}^0(D^\bullet(\mathcal{U})) \to \check{\mathrm{H}}^1(\mathcal{U}, \mathcal{F}) \to 0$$

At this point we can appeal to the fact that we have done the 0 case so we have isomorphisms in the first two terms and a natural map $D^\bullet \to C^\bullet(\mathcal{U}, \mathcal{R})$ so we have the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^0(X, \mathcal{F}) \arrow[r] & \mathrm{H}^0(X, \mathcal{G}) \arrow[r] & \mathrm{H}^0(X, \mathcal{R}) \arrow[r] & \mathrm{H}^1(X, \mathcal{F}) \arrow[r] & 0 \\
0 \arrow[r] \arrow[u, "\cong"] & \check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[u, "\cong"] & \check{\mathrm{H}}^0(\mathcal{U}, \mathcal{G}) \arrow[r] \arrow[u, "\cong"] & \mathrm{H}^0(X, D^\bullet(\mathcal{U})) \arrow[r] \arrow[u, "?"] & \check{\mathrm{H}}^1(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[u] & 0 \arrow[u, "\cong"]
\end{tikzcd}$$

Since we're taking a colimit over a directed set we maintain exactness in the limit so by the five lemma it is sufficient to show that in the limit the map

$$\lim_{\mathcal{U}} D^0(\mathcal{U}) = \lim_{\mathcal{U}} \mathrm{H}^0(X, D^\bullet(\mathcal{U})) \to \mathrm{H}^0(X, \mathcal{R}) = \Gamma(X, \mathcal{R})$$

is an isomorphism. To show that this map is surjective by the definition of the quotient for any $r \in \Gamma(X, \mathcal{R})$, we can find a sufficiently small cover $\mathcal{V} = \{V_i\}$ with elements $g_i \in \Gamma(V_i, \mathcal{G})$ that map down to r and each $g_i - g_j \in \Gamma(V_i \cap V_j, \mathcal{F})$. Clearly then the equivalence class of this (g_i) in $D^0(\mathcal{V})$ map onto r so taking the equivalence class of this in the limit we get something hitting r so we get surjectivity. Then for injectivity if something nonzero maps to zero then there must be some family (g_i) that are not sections of $\mathcal{F}$ so that every refinement maps to zero, this is clearly impossible as by the properties of quotients if something goes to zero for some sufficiently fine covering it maps into sections of $\mathcal{F}$, but this means that it must be zero already in $D^\bullet(\mathcal{V})$ so this map is injective. Hence when we apply the five lemma to our diagram we get an isomorphism so we have isomorphisms in degrees $0, 1$ □

And so via this Čech cohomology we can prove Theorem 5.2.2

Proof. Given a line bundle $\mathcal{L}$ and a covering $\mathcal{U}$ that trivialises $\mathcal{L}$ we have on each U an isomorphism $\mathcal{L}(U) \to \mathcal{O}_X(U)$ can construct an element on

$$C^1(\mathcal{U}, \mathcal{O}_X^\times) = \prod_{i<j} \mathcal{O}_X^\times(U_i \cap U_j)$$

Via the map

$$\begin{tikzcd}
\mathcal{L}(U_i) \arrow[r, "\sim"] \arrow[d, "res"] & \mathcal{O}_X(U_i) & s \arrow[r, mapsto] \arrow[d, mapsto] & 1 \\
\mathcal{L}(U_i \cap U_j) \arrow[r, "\sim"] & \mathcal{O}_X(U_i \cap U_j) & s|_{U_i \cap U_j} \arrow[r, mapsto] & \check{s}
\end{tikzcd}$$

This map $\mathcal{L} \mapsto \check{s}$ since by bilinearity of $\otimes$ $\mathcal{L} \otimes \mathcal{L}' \mapsto s \check{\otimes} s' = \check{s}\check{s}'1 \check{\otimes} 1 = \check{s}\check{s}'$ It suffices to show that this map is surjective with homology and injective. This map is injective since if $\mathcal{L} \to 1$ for everything in a cover then by gluing there must be some global section S that restricts to 1 everywhere, that is $\mathcal{L}$ has a non vanishing global section so is trivial so $\mathcal{L} \mapsto 1 \implies \mathcal{L} = \mathcal{O}_X$. It is then surjective since if $s \to 0$ under δ is the same as saying that the difference of the triple intersections is zero, ie the restriction to the triple intersection is well defined. This is precicely the cocycle condition which is true for vector bundles so this map is well defined for homology. Additionally since such a section is well defineed on these intersections we can construct the line bundle so that each fiber is $\mathcal{O}_x \cdot s_{ij}$ on $U_i \cap U_j$, thus we have surjectivity so the map $\mathrm{Pic}(X) \to \check{\mathrm{H}}(X, \mathcal{O}_X^\times) \to \mathrm{H}(X, \mathcal{O}_X^\times)$ is in fact an isomorphism. $\square$

5.3 The Picard Functor(s)

The fact that this $\mathrm{Pic}(X)$ is such a well behaved object that parameterises the isomorphism classes should inspire one to consider it as a moduli space. The issue is that it is not a space. So we want to do the normal thing and consider $\mathrm{Pic}(X)$ as the $X-$points of some scheme. To solve this problem we first must formulate it correctly. The first problem we run into is that Pic has no chance of being a sheaf so definitly isn't representable[FGAE], to try to fix this we can generalise slightly to looking at line bundles over a base

Definition 5.3.1 (*Absolute Picard functor*)**.** For an S-scheme $f : X \to S$ we define

$$\mathrm{Pic}_X(T) = \mathrm{Pic}(X \times_S T) = \mathrm{Pic}(X_T)$$

The issue is that this functor is basically as strong as the last one, and so is too not representable [FGAE] To avoid this then we consider the relative picard functor

Definition 5.3.2. For an S-scheme $f : X \to S$ we define

$$\mathrm{Pic}_{X/S}(T) = \mathrm{Pic}(X_T)/\,\mathrm{Pic}(T)$$

Which can be made into a sheaf without much adjustment by taking the sheafification (see Appendix A) (in fact sometimes it is a sheaf already)

We then say that if the relative Picard functor or any of its sheafifications are representable, the representing scheme is the Picard scheme called $\mathrm{Pic}_{X/S}$. For this there is a very strong classification theorem for when it exists

Theorem 5.3.3 (Existence of the Picard Scheme (Grothendieck))**.** *[FGAE] If X/S is projective, flat with integral geometric fibers, then the étale relative Picard functor* $\mathrm{Pic}_{X/S,\acute{e}t}$ *is representable.*

Proof. This proof is *difficult* but here we give the outline [BEJP]

1. Find a comparison between Cartier divisors and line bundles
2. Prove that the moduli functor of relative Cartier divisors is representable by some open subscheme of the Hilbert scheme
3. Using the comparison find a morphism of the relative functors
4. Adjust the map so that it represents a quotient of the functor of relative cartier divisors by a proper, smooth equivalence
5. Show that in general, such a quotient is a scheme

□

5.4 The Picard Stack

One issue with the representability problem is that it is often the case that the picard functor is not representable. This can be explained with spectral sequences (see [LABP]) however it is enough for us to know that this picard scheme, while powerful, will not work always.

This is one of the ways that stacks are incredibly useful. They allow us to define a very naiive approach to the representability problem and just check when that object is geometric in nature, be that algebraic, Deligne-Mumford etc. Following this we define the following

Definition 5.4.1 (Picard Stack)**.** For a scheme S and a morphism $\pi : X \to B$ over S. We define the stack $\mathcal{P}ic_{X/B}$. The picard stack, as the following functor.

1. For a scheme U The objects of $\mathrm{Pic}_{X/B}(U)$ are pairs $(b, \mathcal{L})$ where
   (a) $b : U \to B$ is a morphism over S
   (b) $\mathcal{L}$ is an invertable sheaf/line bundle on the base change $X_U = U \times_{b,B} X$
2. The morphisms of $\mathrm{Pic}_{X/B}(U)$ are isomorphisms of line bundles

3. For a morphism $f : U' \to U$, $\mathcal{P}ic_{X/B}(f)$ is the functor sending $(b, \mathcal{L}) \mapsto (b \circ f, f^*\mathcal{L})$ with morphisms and sending an isomorphism ϕ to just $f^*\phi$

This stack will obviously solve the representability problem as it is just constructed to do so, the magic is that this stack is in fact often algebraic.

Theorem 5.4.2. *If $\pi : X \to B$ is flat, of finite presentation, and proper, then $\mathcal{P}ic_{X/B}$ is an algebraic stack*

Proof. See [STAC] Proposition 99.10.2. The proof appeals to the representability of the natural map $\mathcal{P}ic_{X/B} \to \mathcal{C}oh_{X/B}$, the stack of coherent sheaves. There isn't time to discuss this stack so the proof is ommitted □

This stack comes equiped with a group structure as the line bundles are invertable so we have a monoidal structure given by $\otimes$, this has natural associator given by simply rebracketing (as most associators are) and so this $\mathcal{P}ic_{X/B}$ is a very easy to work with algebraic group stack. In fact since the $\otimes$ map is commutative it is an abelian group stack

5.5 Rigidification

Since this stack is meant to solve some problems and generalise the standard $\mathrm{Pic}_{X/B}$ scheme, we should hope that we can recover this scheme from our stack. One way to do so is to find what's called the course moduli scheme or rigidification of this stack.

The thing that stops the $\mathcal{P}ic(X)$ from being a scheme is that is contains the information of automorphisms that clearly cannot show up in a set. Luckily there is a way to remove this action

Theorem 5.5.1. *[ACVI] For $\mathcal{X}$ an algebraic stack over S and H a flat, finitely presented, separated group scheme over S. If for every object $p \in \mathcal{X}(T)$ there is an embedding $H(T) \hookrightarrow \mathrm{Aut}_{\mathcal{X}(T)}(p)$ which is compatible under pullbacks, in the sense that for any arrow $\phi : p \to p'$ iver $f : T \to T'$ and $g \in H(T')$, we have that $g \circ \phi = \phi \circ f^*g$. Then there exists an algebraic stack $\mathcal{X}/H$ and a morphism $\rho : \mathcal{X} \to \mathcal{X}/H$ which is an fppf gerbe[1] so that for any $p \in \mathcal{X}(T)$, the morphism $\mathrm{Aut}_{\mathcal{X}(T)}(p) \to \mathrm{Aut}_{\mathcal{X}/H(T)}(p)$ is surjective with kernel $H(T)$*

In the special case of this picard stack the automorphisms we must remove correspond to the action of the multiplicitive group and so when we take $\mathcal{P}ic$ and "remove" $\mathbb{G}_m$ we get Pic

Theorem 5.5.2. *[HOER] For a curve X over k, the picard stack $\mathcal{P}ic(X)$ is representable and we have an exact sequence*

$$B\mathbb{G}_m \to \mathcal{P}ic_k(X) \to \mathrm{Pic}_k(X)$$

[1] This is just a stack for whom any two objects in $F(U)$ are locally isomorphic, for example our stacks of principal G bundles BG are all locally isomorphic to U

Where $\operatorname{Pic}_k(X)$ *is the quotient*

In addition if X *has a k-rational point then this sequence splits so*

$$\mathcal{P}ic_k(X) = B\mathbb{G}_m \times \operatorname{Pic}_k(X)$$

And so from this natural construction of a stack we see that we can recover the picard scheme by just removing the obvious automorphisms

6.1 Conclusion

In conclusion, through the use of the Picard stack and quotient stacks we are able to understand how the foundations in category theory from Chapter 1 and Chapter 3 allow us to define powerful general objects that can be applied in cases where standard scheme theoretic geometry fails. In addition the description of group stacks concretely as functors taking values in 2-groups allows us to get a very solid hold on, for example, the quotient stack of two group schemes. While there has not been space to discuss in this document, the theory of 2-groups is well understood with its own representation theory [LORE] and interpretations as crossed modules [BROW]. This means that using what we have learned here, interpreting the quotients of group schemes and the picard group as stacks, we can use these techniques to better understand group schemes and algebraic geometry as a whole.

Appendix A

Sheafification

Occasionally we will run into functors that are not sheaves. For this it is useful to find a sheaf that is as close as you can get. In the classical case this sheafification is reasonably straightforward. You can just take sections of the space $\bigsqcup \mathcal{F}_x$ which has the topology that makes it all work. This is however quite dependant on stalks which are not as plentiful once we move up to sites so some work is required to sheafifiy correctly

Theorem A.0.1 (Sheafification). *For a site $\mathcal{C}$. The inclusion $\mathbf{Sh}(C) \hookrightarrow \mathbf{Psh}(C)$, that is the category of sheaves into the category of presheaves, admits a left adjoint $((-)^+)^+$*

The standard way to sheafify is to first separate the preshef and then sheafify this separated presheaf

Definition A.0.2. A presheaf $\mathcal{F}$ is separated if the natural map $\mathcal{F}(U) \to \prod \mathcal{F}(U_\lambda)$ is injective

Recall from Section 5.2 we defined the Čech cohomology associated to a cover $\mathcal{U}$ of the space. This naturaly generalises to a site, this allows us to look at the Čech cohomology at an element U of a site by considering the category of coverings of U ordered by refinemenent, call it $\mathcal{J}_U$ and then taking

$$\check{H}^i(U, \mathcal{F}) = \varinjlim_{\mathcal{J}_U^{op}} \check{H}^i(\mathcal{U}, \mathcal{F})$$

As it turns out the functor $F^+(-) = \check{H}^0(-, \mathcal{F})$ acts like the sheafification we want, note that this comes with a natural map induced by each of the maps $\mathcal{F}(U) \to \prod_{\lambda \in \Lambda} \mathcal{F}(U)_\lambda$ for any covering of U

Theorem A.0.3. *1. F^+ is always separated*

2. If $\mathcal{F}$ is separated then $\mathcal{F}^+$ is a sheaf and $\mathcal{F} \to \mathcal{F}^+$ is injective

3. If $\mathcal{F}$ is a sheaf then $F \to \mathcal{F}^+$ is an isomorphim

So the double application is a sheaf

$$\begin{tikzcd} \textit{Presheaf} \arrow[r, "+"] & \textit{Separated Presheaf} \arrow[r, "+"] & \textit{Sheaf} \arrow[loop below, "+"] \end{tikzcd}$$

Additionally any map $\mathcal{F} \to \mathcal{G}$ *for a sheaf* $\mathcal{G}$ *factors through the canonical map* $\mathcal{F} \to \mathcal{F}^{++}$*, that is this sheafification is an adjoint to the inclusion*

Proof. See [STAC] Theorem 7.10.10 and Proposition 7.10.12 □

A similiar process can be done to stacks called "stackification" making the category of stacks a reflexive subcategory of the category of prestacks

Appendix B

Grothendieck Construction

For those who have read a little about stacks this document may seem nonstandard since we take our stacks to be functors $\mathcal{C}^{op} \to \mathbf{Gpd}$ plus descent instead of fibered categories $X : S \to \mathcal{C}$ where each preimage $X^{-1}(p)$ is a groupoid, plus descent.

To be clear what I mean by this

Definition B.0.1 (*Fibered Category*)**.** Suppose we have some category over $\mathcal{C}$, $F : \kappa \to \mathcal{C}$ then we say that κ is fibred over C if for any c there is a so called pullback $f : a \to c$ so that for any $g : b \to c$ with $F(f) = F(g) \circ H$ then there is a unique lift h so that $F(h) = H$. We write $x = F^* y$, since this is defined universally x is unique up to isomorphism

It is better summarised in the diagram

$$\begin{tikzcd}
a \arrow[rr, "f"] \arrow[dr, dashed, "h"] \arrow[dd, mapsto] & & c \arrow[dd, mapsto] \\
& b \arrow[ur, "g"'] \arrow[dd, mapsto] & \\
F(a) \arrow[rr, "Ff"] \arrow[dr, "Fh"'] & & F(c) \\
& F(b) \arrow[ur, "Fg"'] &
\end{tikzcd}$$

Definition B.0.2 (*Stacks by Fibred Categories*)**.** The definition of a stack is then just the defition from Theorem 1.2.6. Ie a stack is a fibred category so that

1. For each $\{U_\lambda \to U\}_{\lambda \in \Lambda}$ and a collection of object $\{X_i\}_{\lambda \in \Lambda}$ with isomorphisms

$$\phi_{\lambda\mu} : X_i|_{U_\lambda \times_U U_\mu} \xrightarrow{\sim} X_j|_{U_\lambda \times_U U_\mu}$$

   that satisfy the cocycle condition $\phi_{\lambda\mu} \circ \phi_{\mu\nu} = \phi_{\lambda\nu}$ on $U_\lambda \times_U U_\mu \times_U U_\nu$. There is some X with isomorphisms on each U_λ, $\phi_\lambda : X|_{U_\lambda} \tilde{\to} X_\lambda$.Here restrictions correspond to the pullbacks of the fibred category.

2. The functor $h(x,y) : \mathcal{C}/U \to \mathbf{Set}$ sending $[F : V \to U] \mapsto \hom(F * x, F * y)$ is a sheaf

These constructions are however very naturally the same. For a functor $\mathcal{C}^{op} \to \mathbf{Cat}$, in our case we have functors taking values in $\mathbf{Gpd} \subset \mathbf{Cat}$, we can convert it into a category fibered over $\mathcal{C}$. The construction is analagous to turning a family of sets indexed by I, $\{A_i\}_{i\in I}$, ie a function $I \to \{A_i\}_{i\in I}$ to a projection $\bigsqcup_{i\in I} A_i \to I$ by $\pi(x \in A_i) = i$, this clearly maintains the same data as our origional indexing, for our functors we want to do the same thing

Definition B.0.3 (*Grothendieck Construction*)**.** For a functor $\mathcal{C}^{op} \to \mathbf{Cat}$ the corresponding fibered category $\int_\mathcal{C} F$ consists of the following

1. The objects of $\int_\mathcal{C} F$ are pairs (x, y) for x an object of $\mathcal{C}$ and y an object of $F(x)$
2. A morphism in $\int_\mathcal{C} F$ that maps $(x, y) \to (x', y')$ is a pair of morphisms $(f : x \to x', \varphi : F(f)(y') \to y)$

This has natural projection $[(f, \varphi) : (x, y) \to (x', y')] \mapsto [f : x \to x']$. This construction is doing basically nothing, we are just taking our disjoint union of $F(c)$ and then adding any morphism that is induced by F

It is mostly immaterial wether we work with the functor $F : \mathcal{C}^{op} \to \mathbf{Cat}$ or the fibered category $\int_\mathcal{C} F \to \mathcal{C}$. Throuought we use the functorial perspective as it is a more natural generalisation of sheaves which are the jumping off point for stacks.

Appendix C

Infinity Categories

This is not meant as a good introduction to infinty categories, it just feels neccecary to mention since in quotients the proof is slightly nicer with these in mind better introductions can be found in [KHAN] or [KERO]. Additionally they can be used to build far more useful intution for n-groups than standard algebra would allow.

We start with the notion of a simplicial set. A simplicial set is the data required to make a simplical complex and thus its data is that of face maps.

Definition C.0.1. The category Δ is the category whos objects are sets of the form $\{0, ..., n\}$ and whos morphisms are just order preserving maps.

Note that any morphism $[k] \to [\ell]$ can be decomposed into a composition of "face maps"

$$\delta_n^i : [n-1] \to [n] \quad j \mapsto \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}$$

And "degeneracy" maps

$$\sigma_n^i : [n+1] \to [n] \quad j \mapsto \begin{cases} j & j \leq i \\ j-1 & j > i \end{cases}$$

Which is precicely the topological data we need for simplicial complexes. And so a simplicial set is just an assignment of these morphisms

Definition C.0.2. A simplicial set is a functor $\Delta^{op} \to \mathbf{Set}$

This is generally written with just the face maps so denoted

$$... \begin{smallmatrix}\to\\\to\\\to\\\to\end{smallmatrix} X_2 \begin{smallmatrix}\to\\\to\\\to\end{smallmatrix} G \times X_1 \rightrightarrows X_0$$

By the Yoneda Lemma for any simplicial set X

$$\hom(\hom(-,[n]),X) \cong X([n]) = X_n$$

So an n simplex of X is just a map $\hom(-,[n]) \to X$. Because of this we refer to $\hom(-,[n])$ as the "standard n simplex" and denote it Δ^n. This is all nice and topological but where do categories come into this. Recall that in topology simplices come with a direction. And so if we have two 1-simplices f, g where the start of one is the end of another we say we have a composition if there is a 1-simplex h where f, g, h are the edges of some 2-simplex. In fact more generally we can take the "horn" Λ^n_k that is the union of all of the faces of the standard n simplex Δ^n excluding the k'th. Our unfilled fg is then given by a morphism $\Lambda^2_1 \to X$, then filling it in corresponds to this morphism factoring through Δ^2. So for higher morphisms we say for a morphism $\sigma : \Lambda^n_k \to X$ for $0 < k < n$ a composition of σ is a morphism $\Delta^n \to X$ such that σ factors as $\Lambda^n_k \hookrightarrow \Delta^n \to X$ for general k this such map is called a fill. We can see that if we just restrict ourselves to the standard world of 1-simplices every horn having composition corresponds to the standard composition laws of category theory and so this is how we define an ∞-category

Definition C.0.3. An ∞-category is a simplicial set $\mathcal{C}$ where every horn $\Lambda^n_k \to \mathcal{C}$ for $0 < k < n$ has a fill

If we allowed for every horn to have a fill, we would have fills for diagrams like

$$\begin{tikzcd} & \bullet \arrow[dr, dashed] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "id"'] & & \bullet \end{tikzcd}$$

And so we would have inverses to every morphism, in fact if we have inverses for every morphism then we get every fill and so we define infinity groupoids by this property

Definition C.0.4. An ∞-groupoid is a simplicial set $\mathcal{G}$ where every horn $\Lambda^n_k \to \mathcal{G}$ for $0 \le k \le n$ has a fill, note that the singular simplicial set of a topological space fills this requirement since it doesnt care about the direction.

In this world we can extend our standard sheaf or stack diagram to form ∞-stacks, using the definitions from [KHAN]

Definition C.0.5. An ∞-Stack on a site $\mathcal{S}$ is a morphism of simplicial sets, or just called a functor, $\mathcal{F}$

$$\mathcal{F} : \mathcal{S} \to \mathbf{Gpd}_\infty$$

So that the diagram

$$F(U) \to \prod_{\lambda \in \Lambda} F(U_\lambda) \rightrightarrows \prod_{\lambda,\mu \in \Lambda} F(U_\lambda \times_U U_\mu) \Rrightarrow \prod_{\lambda,\mu,\nu \in \Lambda} F(U_\lambda \times_U U_\mu \times_U U_\nu)$$

is a limit

To go further than this one would require the infinitely long sheaf diagram, but for us there is no need since we would then be tempted to talk about how to generalise sites and this will become far more technical than we want.

Bibliography

[YUAN] Qiaochu Yuan (https://math.stackexchange.com/users/232/qiaochu-yuan). *Geometrically, why do line bundles have inverses with respect to the tensor product?* Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/1290676 (version: 2020-06-12). eprint: `https://math.stackexchange.com/q/1290676`. URL: `https://math.stackexchange.com/q/1290676`.

[ELEN] Georges Elencwajg (https://math.stackexchange.com/users/3217/georges-elencwajg). *Line bundles of the circle*. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/531547 (version: 2016-02-05). eprint: `https://math.stackexchange.com/q/531547`. URL: `https://math.stackexchange.com/q/531547`.

[BEJF] Dori Bejleri (https://mathoverflow.net/users/12402/dori-bejleri). *The fibre product of two quotient stacks*. MathOverflow. URL:https://mathoverflow.net/q/152738 (version: 2013-12-24). eprint: `https://mathoverflow.net/q/152738`. URL: `https://mathoverflow.net/q/152738`.

[ACVI] Dan Abramovich, Alessio Corti, and Angelo Vistoli. *Twisted bundles and admissible covers*. 2001. arXiv: `math/0106211 [math.AG]`. URL: `https://arxiv.org/abs/math/0106211`.

[ARTI] Michael Artin. *Grothendieck Topologies*. 1962.

[FGAE] B.Fantechi-L.Gottsche-L.Illusie-S.L.Kleiman-N.Nitsure-A.Vistoli. *Fundamental Algebraic Geometry, Grothendieck's FGA Explained*. American Mathematical Society, 2005.

[BEJP] Dori Bejleri. *The Picard functor*. URL: `https://people.math.harvard.edu/~bejleri/teaching/math259xfa19/math259x_lecture12.pdf`.

[BROW] Ronald Brown and Christopher B Spencer. "G-groupoids, crossed modules and the fundamental groupoid of a topological group". In: *Indagationes Mathematicae (Proceedings)* 79.4 (1976), pp. 296–302. ISSN: 1385-7258. DOI: `https://doi.org/10.1016/1385-7258(76)90068-8`. URL: `https://www.sciencedirect.com/science/article/pii/1385725876900688`.

[SGA4] Pierre Deligne. "LA FORMULE DE DUALITE GLOBALE". In: *Séminaire de Géométrie Algébrique du Bois Marie* 4.XVIII (1963), pp. 481–587.

[GROE] Michael Groechenig. *Algebraic Stacks.* URL: %5Chref%7Bhttp://individual.utoronto.ca/groechenig/stacks.pdf%7D.

[HOER] Dr. Fritz Hörmann. *Picard stacks and Jacobian stacks of curves.* URL: https://fhoermann.org/picardstacks.pdf.

[KHAN] Adeel A. Khan. *A MODERN INTRODUCTION TO ALGEBRAIC STACKS.* URL: https://www.preschema.com/lecture-notes/2022-stacks/stacksncts.pdf.

[LERM] Eugene Lerman. *Orbifolds as stacks?* 2009. arXiv: 0806.4160 [math.DG]. URL: https://arxiv.org/abs/0806.4160.

[LORE] Fosco Lorengian. *The algebra and geometry of categorical groups.* URL: https://tetrapharmakon.github.io/stuff/repbeamer.pdf.

[KERO] Jacob Lurie. *Kerodon.* https://kerodon.net. 2018.

[LABG] nLab authors. *Grothendieck topology.* https://ncatlab.org/nlab/show/Grothendieck+topology. Revision 48. Jan. 2025.

[LABP] nLab authors. *Picard scheme.* https://ncatlab.org/nlab/show/Picard+scheme. Revision 20. Mar. 2025.

[OLSS] Martin Olsson. *Algebraic Spaces and Stacks.* American Mathematical Society, 2016. ISBN: 9781470428655. DOI: 10.1090/coll/062. URL: http://dx.doi.org/10.1090/coll/062.

[STAC] The Stacks Project Authors. *Stacks Project.* https://stacks.math.columbia.edu. 2018.

[VAKI] Ravi Vakil. *The Rising Sea - Foundations of Algebraic Geometry.* URL: https://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf.