Manifolds

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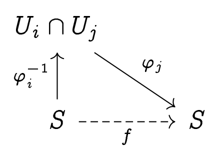
1 Baby's First Constructions

Definition 1.1. An atlas on M compatible with some pseudogroup of transformations[1] Γ on S is a family of pairs $\{(U_i, \varphi_i : U_i \to S)\}$ such that

1. Each U_i is open and the set of all U_i cover M
2. Each φ_i is a homeomorphism onto some open set of S
3. Whenever we have the following diagram



   Then $f \in \Gamma$

We say that an atlas is complete if it isn't contained in any other atlas compatible with Γ

This definition is relatively abstract so we'll limit ourselves to the natural case of $\mathbb{R}^n$ which is what we want to study, spaces that are locally just like $\mathbb{R}^n$

Definition 1.2. A $\Gamma-$manifold is a Hausdorff, second countable topological space M along with a complete atlas compatible with Γ

Definition 1.3. A smooth, $n-$dimensional manifold is a $C^\infty(\mathbb{R}^n)$-manifold

Definition 1.4. We say that f is a morphism or map of manifolds if whenever $f(U_i) \subset V_i$ then

[1] This just means it satisfies natural restrictions, gluings, composition inverses etc, basically a group thats also a sheaf

$$\begin{array}{ccc} U_i & \xrightarrow{f} & V_j \\ \uparrow & & \downarrow \\ S & \xrightarrow{\bar{f}} & S \end{array}$$

$\bar{f} \in \Gamma$

It is alternatively sufficient that for each U_i there is some V_j where this is true as all other V_j will be implied by the transition maps being in Γ
If we have a manifold then since some subsets of $\mathbb{R}^n$ are manifolds like the circle we want to define what it means for some subset to be a manifold so we can inherit the structure in a natural way

Definition 1.5. For M a manifold of dimension d. A subset Y is a submanifold of dimension e if at each point $y \in Y$ there's a chart (U, φ) around y such that

$$\varphi(U \cap Y) = \varphi(U) \cap \mathbb{R}^e \in \mathbb{R}^d$$

Where $\mathbb{R}^e$ is embedded in $\mathbb{R}^d$ in a natural way
Here we say that Y has codimension $c = d - e$

Proposition 1.1. *Submanifolds are manifolds by just taking all charts that satisfy the requirement in the definition and restricting them to Y*

This definition is annoying to check, luckily its a quick consequence of the inverse function theorem that for subsets defined by sufficiently nice functions, they are submanifolds immediately

Proposition 1.2. *For a $d-$dimensional manifold M and $Y \subset M$. If for each $y \in Y$ there is some chart (U_i, φ_i) and smooth functions $f_1 ... f_c$ on $\mathbb{R}^n \supset U' \cong U \to \mathbb{R}$ such that*

$$Y \cap U = \{p \in U | f_1(p) = ... = f_c(p) = 0\}$$

And the determinant

$$\det\left(\frac{\partial f_i}{\partial x_j}(p)\right) \neq 0$$

At each p then Y is a submanifold of codimension c

Proof. We just need to show we can make a choice of coordinates such that $y_i = f_i(p)$ then $Y \cap U$ will be, with respect to these coordinates, starting with a bunch of zeros so will be $\phi(U) \cap \mathbb{R}^e$. To do so we just need to show that this satisfies the transition function stuff. We need only check this with the chart we already have so.

$$\begin{array}{ccc} Y \cap U & = & Y \cap U \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle y} \\ \mathbb{R}^d & & \mathbb{R}^d \end{array}$$

But this transition function is definitionally (f, id) so we just need to show that (f, id) is a local diffeomorphism, this is just inverse function theorem since the jacobian of this is

$$\begin{pmatrix} Df & \text{stuff} \\ 0 & \mathrm{id} \end{pmatrix}$$

So the determinant is nonzero so by the inverse function theorem we are done. □

Ok, thats the nice type of manifold, now for the cruel and evil type, quotient manifolds. Taking quotients is a pain because inheriting the nice smooth structure isn't really that doable[2], see if we take $\mathbb{R}^2/(x \sim -x)$ we get the upper half plane with the two ends of the axis glued together, so we get a sort of spike in the middle so the natural manifold structure wont be inherited nicely because its not smooth (although at least in this case the structure does exist because its topologically just $\mathbb{R}^2$ again). We want to restrict our attention then to sufficiently nice equivalences.

Definition 1.6. G is called a properly discontinuous group of diffeomorphisms of M if for any two compact subsets $K_1, K_2 \subset M$ the set

$$\{g \in G | g(K_1) \cap K_2 \neq \varnothing\}$$

Is finite
We say G has no fixed points if $g(x) = x \implies g = \mathrm{id}$

The reason for this definition is that we want for some small enough set K that every non identity map sends it away from itself so locally $K, g(K)$ are split apart so we have nothing even nearing a fixed point

Proposition 1.3. *For M a manifold and G a properly discontinuous group of diffeomorphisms the space M/G can be given a natural manifold structure*

Proof. We first show the claim that for any q there is some small enough neighborhood U of q such that $gU \cap U \neq \varnothing \implies g = \mathrm{id}$.
(Proof of Claim) Since M is second countable take some base of neighborhoods of q where each is relatively compact[3]

$$U_1 \supset U_2 \supset U_3 \supset \ldots$$

Then each set $F_n = \{gU_n \cap U_n \neq 0\}$ is finite by assumption and

$$F_1 \supset F_2 \supset F_3 \supset \ldots$$

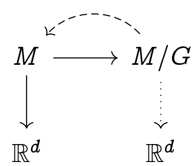
[2]We clearly cant do any quotient like $S^1/\{-1, 1\} \cong S^1 \vee S^1$ so we already want to only look at those quotients that dont really single out points, that is we want to quotient group actions

[3]has compact closure

If each F_i contains some non identity element g_i then by the assumption that they are finite there must be some $\mathrm{id} \neq g \in \bigcap F_i$. Meaning

$$gU_m \cap U_m \neq \varnothing \ \forall m$$

However, now we can choose $x_i \in U_i$ such that $g(x_i) \in U_i$. Then since the U_i base the topology near q, $x_i \to q$ and $g(x_i) \to q$ so $g(q) = q$ which contradicts that there are no fixed points
This shows that we can cover M in open sets U such that for $g \neq \mathrm{id}$ gU and U are disjoint, so for $p_1, p_2 \in U$, if $p_1 \neq p_2$ then their orbits must be distinct since if $gp_1 = hp_2 \implies h^{-1}gp_1 = p_2 \implies h^{-1}gp_1 \in U \implies h^{-1}g = \mathrm{id} \implies g = h$ so $gp_1 = gp_2$ so $p_1 = p_2$ so the map $p \to \mathrm{Orb}(p)$ is locally injective so is locally a homeomorphism onto the image so we can just take the chart maps to be the local inverse composed with the coordinates on M



□

phew that was a pain, but not too bad once you get your head around it. The next section seems to be surgeries but I'm pretty sure we didn't do that which is nice.

2 Baby's Second Constructions

Tangents!! Whoo!! So, when we're dealing in $\mathbb{R}^2$ the tangent to a curve is a vector, well more accurately its a pair of numbers $T = (a, b)$. So reeallly its a function that you give a coordinate system (say the x coordinate) and it tells you how much the curve is moving in that direction (so you could say that $T(\pi_x) = a, T(\pi_y) = b$). On a manifold we don't have it as nice because there isn't a canonical coordinate system, there's lots and lots of different ones so instead of representing the tangent to a curve as a simple vector, we need to tell you how fast its moving with respect to all coordinate systems. We first classify all coordinate systems

Definition 2.1. We define $C^\infty_{M,p} = \mathcal{O}_{M,p}$ the $\mathbb{R}-$algebra of germs of smooth functions defined near p as the set of pairs (f, U) where $f : U \to \mathbb{R}$ is smooth. We take this algebra modulo agreeing on some neighborhood so $(f, U) \sim (g, V)$ if there is some $p \in U' \subset U \cap V$ where $f(x) = g(x)$ for $x \in U'$

If we were to have taken an alternative definition for a manifold at the start as a locally ringed space $(M, \mathcal{O}_M)$ that's locally isomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ then $\mathcal{O}_M$ represents the natural structure sheaf where $\mathcal{O}(U)$ is the set of smooth function $U \to \mathbb{R}$ and so the stalk at p would be precisely this $\mathcal{O}_{M,p}$. Its perhaps a decent exercise to explain why this $\mathcal{O}_{M,p}$ is a local ring making $(M, \mathcal{O}_M)$ a locally ringed space.

Definition 2.2. For some curve $\gamma : [-\varepsilon, \varepsilon] \to M$ where $\gamma(0) = p$ we define the tangent vector to the curve at p as the map

$$X_{\gamma,p} : \mathcal{O}_{M,p} \to \mathbb{R}$$

Where

$$X_{\gamma,p} f = \frac{df(\gamma(t))}{dt}(0)$$

We will often omit the γ, p part as it usually either doesnt matter or is obvious as to which tangent we're talking about

It's worth convincing yourself that this is the correct notion by using our example from before that turns (a, b) into $T(\pi_x) = a, T(\pi_y) = b$

Corollary 2.0.1. *This map satisfies the following conditions we'd expect of a tangent*

1. *X is $\mathbb{R}-$linear*
2. *X satisfies the Leibniz rule*

$$X(f \cdot g) = X(f) \cdot g(p) + f(p) \cdot X(g)$$

This means that $X_{\gamma,p}$ is a derivation at p

Proposition 2.1. *The Tangent Space at p, $T_p(M)$. That is the space of derivations at p that arise as the tangent vector of some curve. Is an n−dimensional vector space. Stronger even, it has a basis given by*

$$\left(\frac{\partial}{\partial x^1}\right)_p, \dots, \left(\frac{\partial}{\partial x^n}\right)_p$$

Where $(U, \varphi = (x^1, \dots, x^n))$ is some chart at p

Proof. First, to make this make sense we define

$$\left(\frac{\partial}{\partial x^i}\right)_p f = X_{\kappa_i,p} f = \frac{d(f \circ \varphi^{-1}(0 \dots t \dots 0))}{dt}(0) = \frac{\partial}{\partial x_i}(f \circ \varphi^{-1})(0)$$

Where κ_i moves along the ith coordinate axis at unit speed then maps up to M by φ. Now for some curve γ, by the chain rule on $\mathbb{R}^n$

$$\begin{aligned} X_\gamma f &= \frac{d}{dt}(f \circ \gamma)(0) \\ &= \frac{d}{dt}(f \circ \varphi^{-1} \circ \varphi \circ \gamma)(0) \\ &= \sum_{i=1}^{n} \left(\frac{d}{dt} x^i \circ \gamma\right) \frac{\partial}{\partial x_i}(f \circ \varphi^{-1})(0) \\ &= \sum_{i=1}^{n} \left(\frac{d}{dt} x^i \circ \gamma\right)(0) \left(\frac{\partial}{\partial x^i}\right)_p f \end{aligned}$$

So any tangent vector can be written as a sum of these
To show that then any sum of these is a tangent vector we just consider that for some linear combination

$$\sum_j \xi_j \left(\frac{\partial}{\partial x^j}\right)$$

We can construct the curve $x^j(\gamma(t)) = x^j(p) + \xi t$ then plugging this into the formula from above gives us what we want □

It is known (I'm not sure if by us) that for a smooth manifold $T_p(M)$ is the space of *all* derivations

Definition 2.3. A vector field V on a manifold M is an assignement of a tangent vector to each point p. We say that such a vector field is smooth if for any smooth function f. $Vf(p) := V(p)f$ is a smooth function. We call the space of smooth vector fields on M $\mathfrak{X}(M)$

This space has some nice properties, by comparing terms on the intersection we can see that for two overlapping neighborhoods $(U_\lambda, y^1...y^n)$ and $(U_\mu, z^1...z^n)$ on the intersection the $\xi_k^{(\lambda)}$ must satisfy

$$\xi_j^{(\lambda)} = \sum_k \frac{\partial y^j}{\partial z^k} \xi_k^{(\mu)}$$

Kinda gross but whatever. More interestingly we can give it a lie algebra strucure with a commutator braket

$$[X, Y]f = X(Yf) - Y(Xf)$$

Expanding this we get some second order derivatives that cancel so in order to ignore wether they exist we just take the definition for

$$X = \sum_j \xi_j \frac{\partial}{\partial x^j} \quad Y = \sum_j \eta_j \frac{\partial}{\partial x^j}$$

$$[X, Y]f = \sum_{j,k} \left(\xi_k \frac{\partial \eta_j}{\partial x^k} - \eta_k \frac{\partial \xi_j}{\partial x^k} \right) \frac{\partial f}{\partial x^j}$$

Being a commutator then, $[\cdot, \cdot]$ is bilinear and for any X, Y, Z

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$$

A computation shows that

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$$

Definition 2.4. For a smooth map of manifolds $f : M \to N$ we define the differential

$$f_* : T_p(M) \to T_{(f(p))}(N)$$

as the map that takes a tangent vector $X_{\gamma,p}$ to the tangent to the curve $f(\gamma(t))$ at $f(p)$. We call this map $(f_*)_p, df_p, Df_p$

By looking at local coordinates[4] we can check that this f_* is just the Jacobian when f is seen as a map $\mathbb{R}^n \to \mathbb{R}^m$

$$\left(\frac{\partial f_i}{\partial x^j}(p) \right)$$

Under our standard basis. Additionally $f_*(X_\gamma)g = X_{f(\gamma)}g = \frac{d}{dt}(g \circ f \circ \gamma) = X_\gamma(g \circ f)$
We now have a big theorem about what f_* implies about f

[4]by all the nice linearity stuff we just need to check that the axis curves work

Proposition 2.2. *For $f : M \to N$ a smooth map*

1. *If df_p is injective there is local coordinates $x^1...x^n$ of p and $y^1...y^m$ of $f(p)$ such that*
$$y^i(f(q)) = x^i(q) \text{ for } i = 1, ..., n$$
Essentially, f is locally just an inclusion map

2. *If df_p is surjective there is local coordinates such that*
$$y^i(f(q)) = x^i(q) \text{ for } i = 1, ..., m$$
Essentially, f is locally just a projection map. This means its an open map.

3. *If df_p is bijective then f is locally a diffeomorphism*

Proof.

1. If df_p is injective then for a local system of coordinates $y^1...y^m$ we have functions $y^i \circ f$ which are functions on $M \to \mathbb{R}$ we want to show that we can find some of these to define a local coordinate system on M. By taking some preimages, in terms of some local coordinates w we have $y^i \circ f = f_i(w^1, ..., w^n)$ so df_p injective means the matrix
$$\left(\frac{\partial f_i}{\partial w^j}\right)(p)$$
has rank n so we can choose n of these indeces so that the square matrix
$$\left(\frac{\partial f_{i_j}}{\partial w^j}\right)(p)$$
Is invertable so by the inverse function theorem $y^{i_j} \circ f$ give a local diffeomorphism so can be used as coordinates so we're done.

2. If df_p is surjective then we do basically the same thing, we take this matrix but now the matrix has rank m so we find an invertable submatrix and take the coordinates $x^i = y^{j_i} \circ f$ or $x^i = w^i$ since we dont care too much what happens to them

3. $1 + 2 = 3$

□

Definition 2.5.

1. If df_p is always surjective then we say that f is a submersion

2. If df_p is always injective then we say that f is an immersion of M into N

3. If f is an injective immersion which is a homeomorphism onto its image, then we call it an embedding

3 Ok now baby's doing calculus

We're gonna start doing things analysts care about ew yuck and look at differential equations. If we have a vector field V then a curve is an integral curve of V if, surprise surprise, its derivative is X. That is to say $X_{\gamma,\gamma(t)} = V(\gamma(t))$
Near any point this is just solving an ODE so we can find a unique curve $\gamma(t)$ that is integral for V for $|t|$ small and $\gamma(0) = p$. In particular we solve the differential equation

$$\frac{d\gamma_i}{dt} = \xi_i(\gamma_1(t), ..., \gamma_n(t))$$

(This is derived by just applying both X_γ and $\sum \xi \partial_x$ to the ith coordinate function then this will solve it always by application of the chain rule) To look at all the solutions at once we introduce

Definition 3.1. A $1-$parameter group of diffeomorphisms or a (global) flow on M is a mapping $\Phi : \mathbb{R} \times M \to M$. $\Phi(t,p) =: \Phi_t(p)$ where

1. For each t, Φ_t is a diffeomorphism
2. For $t, s \in \mathbb{R}$ $\Phi_{t+s} = \Phi_t \circ \Phi_s$

This is essentially a map that flows M along the vector field. This flow induces curves at each point where you follow them $\gamma_p(t) = \Phi_t(p)$ which thus induces a vector field $V(p) = X_{\gamma_p(t),p}$ for which these γ are integral curves. Sadly this being well defined for all of $\mathbb{R}$ is too hard in most cases so we resort to just being able to flow M a little bit

Definition 3.2. An (ε)local 1 parameter group of diffeomorphisms is the same as the last definition but instead of $\mathbb{R}$ we have $(-\varepsilon, \varepsilon)$ and then (2.) is true whenever its well defined

This still encapsulates some less nice integral curves and lets us induce a vector field. In fact any vector field can make one of these by just flowing each point as far as it will go.

Proposition 3.1. *Let V be a vector field on M. Then for each point $p \in M$ there is an open neighborhood U and an (ε)local 1-parameter group of diffeomorphisms $\Phi_t : U \to M$ inducing V*
Additionally if two such groups induce the same vector field, they coincide

Proof. We do as before and construct the integral curves of X starting at p, say $\gamma_p(t)$ then we define $\Phi_t(p) = \gamma_p(t)$ thanks to some results about ODEs this varies smoothly with t, p and are unique. So this gives us what we want. 2. follows since they both give valid curves so by uniqueness of ODE solutions they are the same. 1. follows by letting $s = -t$ and perhaps restricting to a smaller open set if needs be □

If this construction ends up giving us a global $1-$parameter group of diffeomorhpisms! Cool! If V does this we say that V is complete. "How often can this happen?" I hear you ask. Well...

Proposition 3.2. *On a compact manifold M, every vector field V is complete*

Proof. We have open neighborhoods for each point where these work, so we cover and take a finite subcover, then these all work on some $\varepsilon = \min\{\varepsilon_i\}$ so we just repeat after time $t = \varepsilon$ to extend it to $\mathbb{R}$. If there are any overlaps they stitch together wonderfully by uniqueness don't even stress □

A good portion of manifolds stuff, like what we just did, is we make something happen locally then extend it to global property. This next tool gives us essentially an algorithm to do that.

Theorem 3.3 (Partitions of Unity)**.** *Let M be a manifold with an open cover $\{U_\lambda\}_{\lambda\in\Lambda}$. Then there exists functions $\{\theta_i\}_{i\in\mathbb{N}}$ on M such that.*

1. *For any p, $0 \leq \theta_i(p) \leq 1$*
2. *Each p has a neighborhood on which all but fintely many θ_i are identically zero*
3. *For each $i \in \mathbb{N}$ there is an index $\alpha(i)$ such that $\operatorname{Supp}\theta_i \subset U_{\alpha(i)}$*
4. *For each p, $\sum_{i\in\mathbb{N}} \theta_i(p) = 1$*

Proof. The proof is very long and not enlightening so im not gonna bother to write it, the cliffnotes are. Obviously you can do this on $\mathbb{R}^n$ since we can make C^∞ functions with compact support. Then we take a compact exaustion[5] of M which exists by second countable-ness. Then for each step of this exaustion and for each new p we havent hit. We intersect with some $U_\alpha \ni p$ and lift our function to one on U_α this gives us an open cover so only finitely many p are needed to have this non zero everywehere. And we're done (we might want to divide through by something to make it 1 but thats neither here nor there) □

We can now prove one of the like, 2 theorems from this module

Definition 3.3. A map of topological spaces is called proper if the preimage of any compact set is compact

Theorem 3.4 (Ehresmann's fibration theorem)**.** *Let $\pi : M \to I = (a, b)$ be a proper submersion of manifolds. Then for any two t_1, t_2 the fibers $\pi^{-1}(t_1), \pi^{-1}(t_2)$ are diffeomorphic*

[5] G_i where $\bar{G}_i$ compact, $\bar{G}_i \subset G_{i+1}$ and $M = \bigcup_i G_i$

Proof. Since π is a submersion it is locally just a projection $\mathbb{R}^n \to \mathbb{R}$ so choose coordinates of a neighborhood $U(p)$ around p such that $\pi(x^1, ..., x^n) = x^1 = t$. Consider the vector field $\frac{\partial}{\partial t}$ on I. On each $U(p)$ there is a vector field $\frac{\partial}{\partial x^1} =: V_{U(p)}$ with

$$d\pi_p \frac{\partial}{\partial x^1} = \frac{\partial}{\partial t}$$

We then choose a partition of unity $\{\theta_i\}$ subordinate to $\{U(p)\}$ so now

$$W = \sum_i \theta_i V_{U(\alpha(i))}$$

Defines a vector field on M with the property that

$$d\pi_p W = \frac{\partial}{\partial t}$$

Let $\Phi : I_\varepsilon \times U \to M$ be the corresponding local 1−parameter group of diffeomorphisms around some point p. If $\pi(p) = 0$[6] then since $d\pi_p W = \frac{\partial}{\partial t}$ this Φ is unit speed so $\Phi(t, p) = p_t$ where $\pi(p_t) = t$

We now take the largest possible interval where this is defined, if this is everywhere we are done since $\Phi(t_0, p) = p_{t_0}$ so $\pi(p_{t_0}) = t_0$ so $p \to \Phi(t_0, p)$ defines a diffeomorphism of fibers with inverse $q \to \Phi(-t_0, q)$. We just need to show that this is defined over all of I

Assume it is not, that is the maximal interval is some $(\omega_1, \omega_2) \subsetneq I$. Wlog assume that $\omega_2 < b$ then we want to show that we can define some value for $\Phi(\omega_2, p)$. We have that $\pi^{-1}([0, \omega_2])$ is compact in M so if we take some increasing sequence $t_k \to \omega_2$ then $\Phi(t_k, p) \in \pi^{-1}([0, \omega_2])$ will have some convergent subsequence converging to some p' therefor we can set $\Phi(\omega_2, p) = p'$ and then at this point we can extend the flow to some $\omega_2 + \varepsilon$ contradicting that the interval was maximal. This means that the maximal interval must be the whole thing so any fiber is diffeomorphic to $\pi^{-1}(0)$ and we are done. □

Essentially the idea is we parameterise a curve such that $\pi(\gamma_p(t)) = t$ then moving along this curve lets us hit everything thanks to $\mathbb{R}$ being nice and ordered and the assumption that π is proper, giving us diffeomorphisms.
That was a pain, luckily its nicer now, because we're just doing geometry

[6]We may assume that $0 \in I$ by just shifting before or after the fact

4 Baby's done with calculus, back to constructions

Definition 4.1. We define the dual tangent space, made up of covectors as $T_p(M)^* = \hom_{\mathbb{R}}(T_p(M), \mathbb{R})$

Definition 4.2. The total differential of a smooth function f at p is the covector satisfying

$$df_p X = Xf \ \forall X \in T_p(M)$$

This coencides with the induced differential f_* where we view f as a map of manifolds and $\mathbb{R} \cong T_{f(p)}(\mathbb{R})$

Definition 4.3. A (smooth) $1-$form ω on M is an assignment of a covector ω_p at each point p such that in a neighborhood of p

$$\omega = \sum_j f_j dx^j$$

Where

$$dx^j : \frac{\partial}{\partial x^j} \to 1, \frac{\partial}{\partial x^i} \to 0$$

And is the total derivative of x^j the coordinate function and where f_j are smooth

We can now construct the tangent/cotangent bundles

Definition 4.4. We define the tangent/cotangent bundles as

$$T(M) = \bigsqcup_{p \in M} T_p(M) \quad T(M)^* = \bigsqcup_{p \in M} T_p(M)^*$$

With the natural manifold structure (natural in the sense that these are locally just the product of M with $\mathbb{R}^n$)

Proof. The general idea for these is we take the natural projection maps

$$\pi : X_p \mapsto p \quad \pi^* : \omega_p \mapsto p$$

Then given a chart $(U, \varphi = (x^1, ..., x^n))$ on M we construct charts on these bundles as

$$(\pi^{-1}(U), \tilde{\varphi}) \quad (\pi^{*-1}(U), \tilde{\varphi^*})$$

Where

$$\varphi(v) = (x^1(\pi(v)), ..., x^n(\pi(v)), dx^1(v), ..., dx^n(v))$$

$$\varphi^*(v) = (x^1(\pi^*(\lambda)), ..., x^n(\pi^*(\lambda)), \lambda\frac{\partial}{\partial x^1}, ..., \lambda\frac{\partial}{\partial x^n})$$

These maps then imply the natural differential and topological structure, where these sets base the topology and these charts form the atlas □

These are examples of a more general class of manifold, the vector bundle

Definition 4.5. A vector bundle of rank r on a smooth manifold M is a smooth manifold E with a smooth map

$$\pi : E \to M$$

Such that there exists an open cover $\{U_j\}$ of M where (U_j, φ_j) are charts with the property that

1. There is a diffeomorphism f_j such that the following commutes

$$\begin{array}{ccc} \pi^{-1}(U_j) & \xrightarrow{f_j} & U_j \times \mathbb{R}^r \\ & \searrow^{\pi} & \downarrow{\scriptstyle \pi_j} \\ & & U_j \end{array}$$

2. For $p \in U_j \cap U_k$, If $(p, x) \in U_j \times \mathbb{R}^r, (p, y) \in U_k \times \mathbb{R}^r$ then

$$f_j \circ f_k^{-1}(p, y) = (p, f_{jk}(p) \cdot x)$$

   Where $f_{jk} : U_j \cap U_k \to \mathrm{GL}(r, \mathbb{R})$ are smooth

These are interesting objects (see $K-$theory) with lots of structure

Definition 4.6. 1. We call $E_p = \pi^{-1}(p)$ the fibre of E over p

2. A morphism of vector bundles $f : E \to F$ is a smooth map of manifolds that maps fibres $E_p \to F_p$ and such that f restricted to $E_p \cong \mathbb{R}^n$ is a linear map of some constant rank s indpendant of p.

3. A (smooth) section in a vector bundle E over an open subset U is a smooth map $\sigma : U \to E$ with $\pi \circ \sigma = \mathrm{id}_U$. In the case of tangent/cotangent bundles these sections are smooth vector fields/ $1-$forms on U

Vector bundles are often constructed from an open cover $\{U_j\}$ smooth functions

$$f_{jk} : U_j \cap U_k \to \mathrm{GL}(r, \mathbb{R})$$

Where (whenever it makes sense to say)

$$f_{jk} = f_{kj}^{-1} \quad f_{jj} = \mathrm{id} \quad f_{ij} \circ f_{jk} = f_{ik}$$

The last condition is called the cocycle condition and a system as just described is called a $\mathrm{GL}(r, \mathbb{R})-$cocycle on M. Clearly any bundle admits a cocycle, but interestingly any cocycle will admit a bundle (the cocycle of E will give back E up to isomorphism)

Definition 4.7. From a cocycle we construct its vector bundle as follows. We take

$$\tilde{F} = \bigsqcup_j U_j \times \mathbb{R}^r$$

then impose the equivalence that for $p \in U_k \cap U_k$, $(p, x) \sim (p, y) \iff x = f_{jk}(p) \cdot y$ then the natural projection maps $(p, x) \to p$ induce the smooth/topological structure as before

We can then define the Whitney direct sum (ooh we're halfway to $K-$theory)

Definition 4.8 (Whitney direct sum)**.** The Whitney direct sum of E, F denoted $E \oplus F$ is the vector bundle of rank $r + s$ associated to the $\mathrm{GL}(r + s, \mathbb{R})$-cocycle with the same cover given by

$$(f + g)_{jk}(p) = \begin{pmatrix} f_{jk}(p) & 0 \\ 0 & g_{jk}(p) \end{pmatrix}$$

5 Baby is getting bored of constructions is there any other content we can do please

Ok so nows the algebra section, im just gonna speedrun it because algebra is easy. We define the tensor product $U \otimes V$ as the vector space spanned by elements $u \otimes v$ with like, the obvious relations that make $(u, v) \to u \otimes v$ bilinear. Its like the "least" information you need to define something bilinear so we have the alternative classification

Proposition 5.1 (Universal Property of Tensor Product)**.** *$U \otimes V$ with the natural map $U \times V \to U \otimes V$ is the unique vector space such that for any bilinear $h : V \times W \to Z$ there is a unique linear map $\bar{h}$ such that the following commutes*

$$\begin{array}{ccc} U \times V & \xrightarrow{\otimes} & U \otimes V \\ & \searrow^{h} & \downarrow \bar{h} \\ & & Z \end{array}$$

Proof. The proof is easy, just find the map and its obviously the only one □

Corollary 5.1.1. *The following isomorphisms/maps unique*

1. $U \otimes V \cong V \otimes U$
2. $k \otimes U \cong U$
3. $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$
4. *Maps* $U_i \to V_i$ *induce maps* $U_i \otimes \to V_i \otimes$
5. $(U_1 \oplus U_2) \otimes V \cong (U_1 \otimes V) \oplus (U_1 \otimes V)$
6. *We have an obvious basis of* $U \otimes V$ *so* $\dim U \dim V = \dim U \otimes V$
7. $U \otimes V \cong \hom(U^*, V)$
8. $U^* \otimes V^* \cong (U \otimes V)^*$

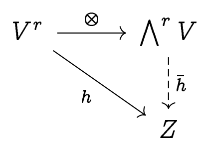
Proof. These mostly follow trivially from the proposition □

Definition 5.1. We define the tensor algebra over a vector space is the free algebra on a vector space, that is we have a forgetful functor from algebras to vector spaces, taking its left adjoint gives us a tensor algebra
An alternative representation is $T^\bullet(V) = \bigoplus V^{\otimes r}$ where $V^{\otimes r}$ is the $r-$fold tensor product. Multiplication is then just given by concatenation

Definition 5.2. We define the exterior algebra as the tensor algebra with anti symmetry $u \wedge v = -v \wedge u$

Definition 5.3. The exterior $r-$forms $\bigwedge^r V$ are given by those elements of degree r its no longer an algebra but who cares

Proposition 5.2. *1. We have the same universal property as before for when h is alternating, there is a unique h so that*



2. *Applying this to the natural map $W^r \to \bigwedge^r W$ a map $V \to W$ induces a map $\varphi : \bigwedge^r V \to \bigwedge^r W$*

3. $\dim \bigwedge^r k^n = \binom{n}{r}$

4. *For $f : k^n \to k^n$ an endomorphism the induced map $\bigwedge^n k^n \to \bigwedge^n k^n$ is multiplication by* $\det f$

5. *There is a natural nondegenerate bilinear pairing $\bigwedge^r V^* \times \bigwedge^r V \to K$ mapping*
$$(v_1^* \wedge \ldots \wedge v_r^*, w_1 \wedge \ldots \wedge w_r) \mapsto \det(v_i^*(w_j))$$
inducing an isomporphism $\bigwedge^r V^* \cong (\bigwedge^r V)^*$

Proof. These are all immediate from the definitions/natural universal properties □

6 Getting comfortable? Too bad! More Calculus!

Definition 6.1. An $r-$form ω on M is a smooth choice of an element in $\bigwedge^r T_p(M)^*$ to each point. Smooth in the sense that in local coordinates

$$\omega = \sum_{1 \leq i_1 \leq \ldots \leq i_r \leq n} f_{i_1 \ldots i_r}(x^1, \ldots, x^r) dx^{i_1} \wedge \ldots \wedge dx^{i_r}$$

Where each $f_{i_1 \ldots i_r}$ is smooth. We call the space of all such forms $\mathcal{A}^r(M)$

Definition 6.2. For $f : M \to N$ the pullback $f^*\omega \in \mathcal{A}^r(M)$ of ω by f is the natural extension of the map

$$(f_*)^* : T_{f(p)}(N)^* \to T_p(M)^*$$

Which is just the dual map of

$$f_* : T_p(M) \to T_{f(p)}(N)$$

In local coordinates we define the map by

$$f^* dy^i = \sum_j \frac{\partial f_i}{\partial x^j} dx^j = df_i$$

Which then extends to $\omega = \sum_I a_j dy^I$ (where I is some multiindex)

$$f^*\omega = \sum_I f^* a_I df_I$$

Where $f^* a_I = a_I \circ f$ and $df_I = df_{i_1} \wedge \ldots \wedge df_{i_r}$

As I'm sure we're all category theorists here the following corollary lets us say that taking the tangent/cotangent space of a pointed manifold gives us a functor $Man_* \to Vect$ or $Vect^{op}$! Yay! It is at this point I feel morally obligated to mention that the chain rule is just functorailty of this and I hope from now on you only refer to the chain rule as functoriality of the derivative to annoy and confuse all your classmates.

Corollary 6.0.1.

1. $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$
2. $f^*(\omega \wedge \theta) = f^*\omega \wedge f^*\theta$
3. $(f \circ h)^*\omega = h^* f^* \omega$
4. $f^*(dy^1 \wedge \ldots \wedge dy^n) = \det\left(\frac{\partial f_i}{\partial x^j}\right) dx^1 \wedge \ldots \wedge dx^n$

The last one when we let f be the transition map we see

$$a(y^1...y^n)dy^1 \wedge ... \wedge dy^n = a(y^1(x)...y^n(x)) \det\left(\frac{\partial y_i}{\partial x_j}\right) dx^1 \wedge ... \wedge dx^n$$

Which looks just like the change of variables formula[7]

$$\int a(y^1...y^n)dy^1 \wedge ... \wedge dy^n = \int a(y^1(x)...y^n(x))|\det\left(\frac{\partial y^i}{\partial x^j}\right)|dx^1 \wedge ... \wedge dx^n$$

Definition 6.3. 1. We say that an $n-$dimensional manifold is orientable if there exist an everywhere nonvanishing $n-$form (called a top dimensional form) ω on it.

2. An orientation on M is a choice of an equivalence class of these forms where $\omega \sim \omega'$ if $\omega = f\omega'$ for some everywhere positive smooth f.

3. An oriented manifold is a pair $(M, [\omega])$ (Clearly there are 2 possible orientations, $[\omega], [-\omega]$)

This definition is however annoyingly technical and doesnt really tell you much about the structure of the manifold, like when can we actually do this? We cant for the Möbius band but we can for a normal band? Whats the difference?

Proposition 6.1. *A manifold M is orientable if and only if there is a covering of coordinate charts such that on each intersection*

$$\det\left(\frac{\partial y^i}{\partial x^j}\right) > 0$$

The idea being. We can clearly orient locally, so if whenever we swap to a new local segment we can choose an orientation that preserves this sign and stuff, then the other direction should just be that an orientation will swap with the sign of this so if it doesn't swap then its positive, lets see how that pans out.

Proof. If M is orientable, oriented by $\omega = f(x_1, ..., x_n)dx^1 \wedge ... \wedge dx^n$ where $f > 0$ on the overlap we have

$$\begin{aligned} g(y^1, ..., y^n)dy^1 \wedge ... \wedge dy^n &= g(y^1(x), ..., y^n(x)) \det\left(\frac{\partial y^i}{\partial x^j}\right) dx^1 \wedge ... \wedge dx^n \\ &= f(x^1, ..., x^n)dx^1 \wedge ... \wedge dx^n \end{aligned}$$

So since $f, g > 0$ then the determinant is positive.
If we have such a covering then we take a partition of unity subordinate to this cover. Then let

$$\omega = \sum_i \theta_i (dy^1 \wedge ... \wedge dy^n)_{\alpha(i)}$$

[7] foreshadowing?

Then on some open set in this cover

$$\omega = \sum_i \theta_i \det\left(\frac{\partial y^i}{\partial x^j}\right) dx^1 \wedge \ldots \wedge dx^n$$

Which is non negative and non vanishing because all $\theta_i \geq 0$ and at least one > 0 and det > 0 ☐

We finally now know enough to do integrals! We define the integral for some oriented manifold and some $n-$form ω with compact support

$$\int_M \omega$$

To do so we choose a covering $\{U_\alpha\}$ such as is in the proposition then scale so on each U_α the orientation is equivalent to $dy^1 \wedge \ldots \wedge dy^n$ then on U_α $\omega = f(y^1, \ldots, y^n)dy^1 \wedge \ldots \wedge dy^n$. Now choosing a partition of unity subordinate to $\{U_\alpha\}$ we have that on U_α

$$\theta_i \omega = g_i(y^1, \ldots, y^n)dy^1 \wedge \ldots \wedge dy^n$$

We now define the integral

$$\int_M \omega := \sum_i \theta_i \omega := \sum_i \int_{\mathbb{R}^n} g_i(x_1, \ldots, x_n)dx_1 \ldots dx_n$$

Whooo!! We did it! Now I know you all know about Stokes' theorem and dont worry we're getting to it. First we need to define a couple more things (I know, more constructions, who could've seen this coming)

Proposition 6.2. *Define $\mathcal{A}(M) = \oplus \mathcal{A}^r(M)$. There is a natural $\mathbb{R}-$linear map $d : \mathcal{A} \to \mathcal{A}$ called the exterior derivative such that*

1. *For an $r-$form ω, $d\omega$ is an $r+1$ form*
2. *For a smooth function $f \in \mathcal{A}^0(M)$ df is what we've already defined*
3. *For an $r-$form ω and $s-$form π*

$$d(\omega \wedge \pi) = d\omega \wedge \pi + (-1)^r \omega \wedge d\pi$$

4. *$d^2 = 0$*

And in coordinates

$$\omega = \sum_I f_I dx^I \implies d\omega = \sum_I df_I \wedge dx^I$$

Proof. By applying a partition of unity it's enough to show that the map we defined locally satisfies these conditions.

1. This is immediate

2. This too is immediate

3. For $\omega = f dx^I, \pi = g dx^J$ we have since this map satisfies the obvious linearity etc.

$$\begin{aligned} d(\omega \wedge \pi) &= d(fg dx^I \wedge dx^J) \\ &= d(fg) \wedge dx^I \wedge dx^J \\ &= (f dg + g df) \wedge dx^I \wedge dx^J \\ &= (-1)^r f dx^I \wedge dg \wedge dx^J + df \wedge dx^I \wedge g dx^J \\ &= (-1)^r \omega \wedge d\pi + d\omega \wedge \pi \end{aligned}$$

4. $$d^2\omega = \sum_j \frac{\partial^2 f}{\partial x^k \partial x^j} dx^k \wedge dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_r}$$

   Now since for each term we add both $\frac{\partial^2 f}{\partial x^k \partial x^j} dx^k \wedge dx^j$ and $\frac{\partial^2 f}{\partial x^k \partial x^j} dx^j \wedge dx^k$ they all cancel so $d^2 = 0$

We now just need to show that this function is well defined, that is to say that if we choose different coordinates does it agree? Well we can derive that it has this form on all coordinate choices just from the properties we now know.

$$d\left(\sum_I f_I dy^I\right) = \sum_I df \wedge dy^I + \sum_I f d(dy^I)$$

Where the second term is zero because

$$d(dy^1 \wedge \dots \wedge dy^k) = d^2 y^1 \wedge \dots \wedge dy^k - dy^1 d(dy^2 \wedge \dots \wedge dy^k)$$

Where the first is zero because $d^2 = 0$ and the second by induction so it is just the first term nomatter what coordinates we choose ☐

Proposition 6.3. *For f a smooth map ω an $r-$form*

$$d(f^*\omega) = f^*(d\omega)$$

Proof. For zero forms this is clear since f_* satisfies $f_*(X)(\varphi) = X(\varphi \circ f)$ so the dual map satisfies $f^*(d\varphi) = d(\varphi \circ f) = d(f^*\varphi)$
Now for a general form

$$\omega = \sum g_I dy^I$$

So

$$f^*\omega = \sum f^*(g_I) f^* dy^I$$

$$d(f^*\omega) = \sum d(f^*(g_I)) f^* dy^I = \sum f^*(d(g_I)) f^* dy^I = f^*(d\omega)$$

□

Corollary 6.3.1. *1. For $f : \mathbb{R}^3 \to \mathbb{R}$ $df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 =$ Grad f*

2. *For $\omega = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$ then $d\omega = \operatorname{curl} f$ (where the 1st coordinate is $dx_2 \wedge dx_3$ and so on)*

3. *For ω a 2 form as a sum of f_i, $d\omega = \operatorname{div} f dx_1 \wedge dx_2 \wedge dx_3$*

4. *So $\operatorname{curl} \operatorname{grad} = 0$ and $\operatorname{div} \operatorname{curl} = 0$ are just $d^2 = 0$*

As an interesting note this can be put into a short(ish) exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(\mathbb{R}^3, \mathbb{R}) \xrightarrow{\text{Grad}} C^\infty(\mathbb{R}^3, \mathbb{R}^3) \xrightarrow{\text{Curl}} C^\infty(\mathbb{R}^3, \mathbb{R}^3)$$

$$\xrightarrow{\text{Div}} C^\infty(\mathbb{R}^3, \mathbb{R}) \longrightarrow 0$$

Theorem 6.4. *For M an oriented $n-$dimensional manifold if $\omega \in \mathcal{A}^{n-1}(M)$ is a differential form with compact support then*

$$\int_M d\omega = 0$$

Ooh starting too looks Stokes' theoremey now aren't we

Proof. Choose a covering and paritition of unity θ to write

$$\omega = \sum_i \theta_i \omega$$

Then locally

$$\theta_i \omega = f_1 dx^2 \wedge dx^3 ... \wedge dx^n - f_2 dx^1 \wedge dx^3 ... \wedge dx^n + f_3 dx^1 \wedge dx^2 ... \wedge dx^n$$

So

$$d(\theta_i \omega) = \left(\frac{\partial f_1}{\partial x^1} + ... + \frac{\partial f_n}{\partial x^n} \right) dx^1 \wedge ... \wedge dx^n$$

So now we're just looking at a sum of integrals of the form

$$\int_{\mathbb{R}^n} \left(\frac{\partial f_1}{\partial x_1} + ... + \frac{\partial f_n}{\partial x_n} \right) dx_1 ... dx_n$$

But

$$\int_{\mathbb{R}^n} \frac{\partial f_1}{\partial x_1} dx_1 ... dx_n = \int_{\mathbb{R}} ... \int_{\mathbb{R}} \frac{\partial f_1}{\partial x_1} dx_1 ... dx_n = \int_{\mathbb{R}} ... [f_1]_{-\infty}^{\infty} ... dx_n = 0$$

Since f_1 has compact support it is eventuall zero on both ends so the overall integral is zero □

We're getting so close I can nearly taste it. Although Stokes' Theorem says something about boundaries, should probably do something about that

Definition 6.4 (Manifold with boundary)**.** A manifold with boundary is a manifold but instead of $\mathbb{R}^n$ we just replace it with

$$H^n = \{(x_1, ..., x_n) \subset \mathbb{R}^n | x_n \geq 0\}$$

Then the pseudogroup is the same just with $C^\infty(H^n)$ and everything. Its all fine

We define the boundary of M as

$$\partial M = \{p \in M | \text{ there is some chart where } \varphi(p) \in \{(x_1 ... x_n) \in \mathbb{R}^n | x_n = 0\}\}$$

I refuse to be more rigourous here, perhaps there has not been a concept yet so intuitive. Although if we want to define integrals on the boundary we need orientations

Proposition 6.5. *If M is an orientable manifold with boundary then ∂M is orientable*

Proof. If $\dim M = 1$ then ∂M is points so this is trivial. For $\dim M \geq 2$. We can choose a cover such that locally the coordinates on intersections have positive transfer determinants, we just need to do this on ∂M we already have the cover, lets check that the determinant stays positive. At a point on the boundary we have

$$y^n(x^1 ... 0) = 0$$

So y^n will be zero nomatter how you change $x^1, ..., x^{n-1}$ so our jacobian is

$$J = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & ... & \frac{\partial y^1}{\partial x^{n-1}} & \frac{\partial y^1}{\partial x^n} \\ \vdots & \ddots & \frac{\partial y^1}{\partial x^{n-1}} & \vdots \\ \frac{\partial y^{n-1}}{\partial x^1} & ... & \frac{\partial y^1}{\partial x^{n-1}} & \vdots \\ 0 & ... & 0 & \frac{\partial y^n}{\partial x^n} \end{pmatrix}$$

And since y^n must increase as x^n increase as they move away from $x_n = 0$ since they can only become postive $\frac{\partial y^n}{\partial x^n} > 0$ so since the determinant of J is positive the top left corner matrix will have positive determinant at $x^n = 0$ so when we restrict these coordinates to x^n the charts give transitions with positive determinant so give an orientation □

Definition 6.5. For an orientable manifold M with orientation $\omega \simeq \kappa dx^1 \wedge ... \wedge dx^n$ for $\kappa = \pm 1$ we define the induced orientation by $(-1)^n dx^1 \wedge ... \wedge dx^{n-1}$

Now we're here, we can do it !! The big one!!

Theorem 6.6 (Stokes' Theorem)**.** *For an oriented manifold with boundary and $\omega \in \mathcal{A}^{n-1}(M)$ with compact support*

$$\int_M d\omega = \int_{\partial M} \omega$$

Proof. For $n = 1$ this is just the fundamental theorem of calculus. So for $n \geq 2$ Let $\omega = \sum_i \theta_i \omega$ and so

$$\int_M d\omega = \sum_i \int_M d(\theta_i \omega)$$

Then writing $\theta_i \omega$ as the annoying alternating sum thing we did earlier

$$\begin{aligned}\int_M d(\theta_i \omega) &= \int_{x_n \geq 0} \left(\frac{\partial f_1}{\partial x_1} + ... + \frac{\partial f_n}{\partial x_n} \right) dx_1 ... dx_n \\ &= \int_{\mathbb{R}^{n-1}} [f_n]_0^\infty dx_1 ... dx_{n-1} \\ &= - \int_{\mathbb{R}^{n-1}} f(x_1, ..., x_{n-1}, 0) dx_1 ... dx_{n-1} \\ &= \int_{\partial M} \theta_i \omega\end{aligned}$$

Where the last equality is due to that $\theta_i \omega = (-1)^{n-1} f_n dx^1 \wedge ... \wedge dx^{n-1}$ and the induced orientation is $(-1)^n dx^1 \wedge ... \wedge dx^{n-1}$ □

Now, to conclude with a nice consequence

Theorem 6.7 (Brouwer's fixed point theorem)**.** *Let B be the closed unit ball in $\mathbb{R}^n$ and $f : B \to B$ a smooth map from B to itself. Then f has a fixed point, (this theorem can be proven for continuous f by using homology but this is cool anyway)*

Proof. Supposing there is no fixed point, $f(x) \neq x$, then we take the line segment $f(x) \to x$ and extend it until it hits the boundary, this induces a smooth map $g : B \to \partial B$ that fixes the boundary. Now take some nowhere vaniguing form on ∂B normalised so

$$\int_{\partial B} \omega = 1$$

Then

$$1 = \int_{\partial B} \omega = \int_{\partial B} g^* \omega = \int_B d(g^* \omega) = \int_B g^*(d\omega) = 0$$

Since $d\omega = 0$ as it's a $n-1$ form on S^n. This is a neat contradiction so f has a fixed point □

References

[1] Damiano Testa Warwick Ma3H5 lecture notes

[2] S. N. Bose National Centre for Basic Sciences lecture notes